Numerical and analytical studies of matched kinetic quasi-equilibrium solutions for an intense charged particle beam propagating through a periodic focusing quadrupole lattice.*

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Alternating-Gradient Transport Systems Use a Spatially Periodic Lattice of Quadrupole Magnets for Transverse Confinement

\[
B'_{q}(z) = B'_q(z)\left(y\hat{e}_x + x\hat{e}_y\right)
\]

\[
F_{foc}(x) = -\kappa_q(z)\left(x\hat{e}_x - y\hat{e}_y\right)
\]

\[
\kappa_q(z) = \frac{ZeB'_q(z)}{\gamma m\beta c^2}
\]
Particle trajectories are fast oscillations on top of slow motion of guiding centers.

\[ \kappa_q(z): \text{FODO lattice} \]

Period \( T_q \) of oscillation center motion

\[ \sigma_v = \frac{\omega_q}{f} \]

\[ \kappa_q(z): \text{sinusoidal lattice} \]
Motivations

- Identifying regimes for quiescent propagation of intense beams over long distances is of the great interest in accelerator research.

- In particular, the development of systematic theoretical approaches that are able to treat self-consistently the applied oscillating force and the nonlinear self-field force of the beam particles simultaneously has been a major challenge of modern beam physics.

- To determine matched-beam quasi-equilibrium distribution functions one needs to determine a dynamical invariant for the beam particles moving in the combined applied and self-generated fields.

- Typically, it is advantageous to eliminate fast oscillations from formalism and describe complex beam particle motion in a new non-oscillating coordinates.

- Standard Hamiltonian techniques are cumbersome due to use of mixed oscillating and non-oscillating independent variables.
Abstract

• New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

• Newly developed technique is specially designed to avoid use of oscillating independent variables. The method is analogous to the Lie transform methods in using only non-oscillating independent variables. At the same time the new approach retains the advantages of simplicity of Hamiltonian methods.

• Using the particle’s vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order $\epsilon^3$.

• The average Hamiltonian is an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

• Making use of this new method equations determining the average self-field potential for general boundary conditions has been obtained for the first time by taking into account the average contribution of the charges induced on the boundary.
References


The transverse dynamics of the intense charged particle beam can be described by the nonlinear Vlasov-Poisson system of equations for the beam distribution function $f(x, p, s)$ and the normalized self-field potential $\Psi(x, s)$.

Here $s = v_b t$ is the longitudinal coordinate, and $v_b$ is the directed beam velocity. The function $f(x, p, s)$ satisfies the nonlinear Vlasov equation

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \sum_{\alpha=1}^{2} \frac{dx^\alpha}{ds} \frac{\partial f}{\partial x^\alpha} + \sum_{\alpha=1}^{2} \frac{dp^\alpha}{ds} \frac{\partial f}{\partial p^\alpha} = 0,$$

where the particle equations of motion are given by

$$\frac{dx^\alpha}{ds} = \frac{\partial H}{\partial p^\alpha}, \quad \frac{dp^\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha},$$

The Hamiltonian $H(x, p, s)$ describes the particle motion in a force field that is the sum of a linear, externally applied, transverse focusing force with components $F^\alpha_{foc} = -\kappa(s) \eta^\alpha x^\alpha$ and the normalized self-field potential $\Psi(x, s)$ is calculated self-consistently using Poisson’s equation.

$\kappa(s)$ is the focusing field strength $\eta^1 = 1$, $\eta^2 = -1$. 
The Hamiltonian $H$ for the particle motion

- It is convenient to introduce the re-normalized variables $\bar{x} = x/a$, $\bar{s} = s/S_0$, $\bar{\kappa}(s) = \kappa(s)/\kappa_0$, $\bar{p} = p/(a\kappa_0 S_0)$ and $\bar{f} = (f/N)a^4(\kappa_0 S_0)^2$, where $S_0$ is the characteristic period of the applied focusing force, $a$ is the characteristic transverse beam dimension, and $\kappa_0$ is the characteristic value of the lattice function $\kappa(s)$.

- After the normalization, the Hamiltonian becomes

$$\bar{H}(\bar{x}, \bar{p}, \bar{s}) = \bar{\kappa}(\bar{s})\left[\eta^\alpha \bar{x}^\alpha \bar{x}^\alpha\right]/2 + \epsilon\left\{\frac{[\bar{p}^\alpha \bar{p}^\alpha]}{2} + \int L(\bar{x}, \bar{x'})\bar{f}(\bar{x'}, \bar{p'}, s')D\bar{x'}D\bar{p'}\right\},$$

where $\epsilon \equiv S_0^2\kappa_0$ and $\int d\bar{x}d\bar{p}\bar{f} = 1$.

- We adopt the notation $[x^\alpha x^\alpha] \equiv \sum_{\alpha=1}^{2} x^\alpha x^\alpha$ and $\int dx_1dx_2Z = \int DxDZ$.

- Green's function $L(\bar{x} - \bar{x'})$ satisfies the equation

$$\left[\frac{\partial}{\partial \bar{x}^\alpha} \frac{\partial}{\partial \bar{x}^\alpha}\right] L(\bar{x} - \bar{x'}) = -s_b\delta(\bar{x} - \bar{x'}).$$

Here, $s_b = 2K/(\kappa_0 S_0)^2a^2 = (4\pi q^2 N/a^2\gamma_b^3)/(\kappa_0 S_0 v_b)^2$ is a measure of the beam space-charge intensity.

- For the beam confined by the external focusing force the maximum value of $(s_b)^{max} \sim 1$. 
Perturbative Hamiltonian Transformation Method

- We search for a time-dependent canonical transformation of the form
  \((x^\alpha, p^\alpha, H, s) \rightarrow (Q^\alpha, P^\alpha, K, s)\)

\[
\begin{align*}
  x^\alpha &= x^\alpha(Q, P, s), \\
  p^\alpha &= p^\alpha(Q, P, s),
\end{align*}
\]

with time-independent transformed Hamiltonian \(K(Q, P)\).

- For every canonical transformation there is a function \(S\) that satisfies the differential relation

\[
[p^\alpha dx^\alpha] - Hds = dS + [P^\alpha dQ^\alpha] - Kds.
\]

- Express \(S = U + p_0(Q, P, s)^\alpha(x - Q)^\alpha\), where \(U(Q, P, s)\) and \(p_0(Q, P, s)\) are functions of the new phase-space variables.

\[
\begin{align*}
  \left( x - Q \right)^\alpha \frac{\partial p_0^\alpha}{\partial P^\beta} &= \left( p - p_0 \right)^\alpha \frac{\partial(x - Q)^\alpha}{\partial P^\beta} - \frac{\partial U}{\partial P^\beta}, \\
  (p - P)^\beta &= - \left( p - p_0 \right)^\alpha \frac{\partial (x - Q)^\alpha}{\partial Q^\beta} + \frac{\partial U}{\partial Q^\beta} + \left( x - Q \right)^\alpha \frac{\partial p_0^\alpha}{\partial Q^\beta}, \\
  K - H &= -(p - p_0)^\alpha \frac{\partial(x - Q)^\alpha}{\partial s} + \frac{\partial U}{\partial s} + (x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial s}.
\end{align*}
\]
The distribution function in the new coordinates

- Particle conservation in the phase-space volume $DxDp$ under the transformation

$$F(Q, P, s) \, DQDP = f(x, p, s) \, DxDp.$$ 

- For a canonical transformation, the phase-space volume is conserved according to $DxDp = DQDP$, and therefore $F(P, Q, s) = f[x(Q, P, s), p(Q, P, s), s]$.

- The new distribution function satisfies the Vlasov equation $dF/ds = 0$.

- For a time-independent Hamiltonian, there exists a trivial solution to the Vlasov equation, $F = G[K(Q, P)]$ for arbitrary function $G$.

- The matched solution can be found from $f(x, p, s) = G\{K_G[K_G(x(p, p, s), P_G(x(p, p, s))]\}$.

- For solutions of this form, the Hamiltonian becomes

$$H(x, p, s) = \frac{\kappa(s)[\eta^\alpha x^\alpha x^\alpha]}{2} + \epsilon\left\{\frac{[p^\alpha p^\alpha]}{2} + \right\} + \int L[x, x(Q, P, s)]G[K(Q, P)]DQDP.$$
Iterative procedure of finding the canonical transformation in terms of the small parameter $\epsilon \sim \sigma_v / 2\pi \ll 1$

- Make an expansion

$$p = p_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n p_n(Q, P, s),$$

$$x = Q + \sum_{n=1}^{\infty} \epsilon^n x_n(Q, P, s),$$

$$U = U_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n U_n(Q, P, s),$$

$$K = K_0(Q, P) + \sum_{n=1}^{\infty} \epsilon^n K_n(Q, P),$$

where $p_n, x_n, U_n$ and $K_n (n = 0, 2, \ldots)$ are functions to be determined by the iterative procedure.

- We expand the function $H$ according to

$$H(x, p, s) \equiv H[Q + \sum_{n=1}^{\infty} \epsilon^n x_n, p_0 + \sum_{n=1}^{\infty} \epsilon^n p_n, s]$$

$$= H_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n H_n(Q, P, s).$$
Iterative procedure up to second order in 
\[ \epsilon \sim \frac{\sigma_v}{2\pi} \]

- The average Hamiltonian \( K \) has the form 
  \[ K = \epsilon(K_1 + \epsilon K_2 + \epsilon^2 K_3 + \cdots) \].
  The \( \epsilon \) in front of the bracket renormalizes the time scale, so that 
  the average dynamics occurs on the slow time-scale \( Q = Q(\epsilon s) \) and 
  \( P = P(\epsilon s) \).

- Therefore, to determine the trajectories \( x(s) \) and \( p(s) \) valid to second 
  order in \( \epsilon \), we need to determine the average Hamiltonian \( K \) valid up to 
  the third order in \( \epsilon \).
Canonical transformation up to second order in $\epsilon \sim \sigma_\nu/2\pi$

\[
x^\alpha = Q^\alpha - \epsilon \kappa^{(2)} \eta^\alpha Q^\alpha + \epsilon^2 \left\{ 2\kappa^{(3)} \eta^\alpha P^\alpha + (\kappa \kappa^{(2)})^{(2)} Q^\alpha \right\},
\]

and

\[
p^\alpha = \left\{ P^\alpha - \kappa^{(1)} \eta^\alpha Q^\alpha \right\} + \epsilon \left\{ \kappa^{(2)} \eta^\alpha P^\alpha + (\kappa \kappa^{(2)})^{(1)} Q^\alpha \right\} + \\
+ \epsilon^2 \left\{ (3 < (\kappa^{(2)})^2 > - 2(\kappa \kappa^{(3)})^{(1)} - (\kappa \kappa^{(2)})^{(2)}) P^\alpha \\
+ (\kappa^{(3)} < (\kappa^{(1)})^2 > - (\kappa \kappa^{(2)})^{(1)}) \eta^\alpha Q^\alpha \right\} + \epsilon \frac{\partial}{\partial Q^\alpha} \psi(Q)^{(1)}.
\]

- Here, $< A > \equiv (1/S) \int_{s_0}^{s_0 + S} d\bar{s} A(\bar{s})$, and $\ll A \gg \equiv A - < A >$. Here, we also introduce the notation $A^{(0)} \equiv \ll A \gg$ and

\[
A^{(n)} \equiv \ll \int ds A^{(n-1)} \gg, \text{ for } n \geq 1
\]

- The time-independent Hamiltonian correct to the third order in $\epsilon$ is

\[
K = \epsilon \left\{ \frac{[P^\alpha P^\alpha]}{2} \left( 1 + 3\epsilon^2 < (\kappa^{(2)})^2 > \right) + \frac{[Q^\alpha Q^\alpha]}{2} \left( < (\kappa^{(1)})^2 > \\
+ \epsilon^2 < ((\kappa \kappa^{(2)})^{(1)})^2 > \right) + < \psi(Q) > \right\}
\]

\[
\psi(Q) = \int D\bar{Q} L(Q^\alpha(1 + \epsilon \eta^\alpha \kappa^{(2)}), \bar{Q}^\alpha(1 + \epsilon \eta^\alpha \kappa^{(2)})) n(\bar{Q}), \quad n(Q) = \int DP F(Q, P).
\]
Approximate vs. Exact

- The approximate expression for the square of the vacuum phase advance $\sigma_v^2$ valid up to forth order in small parameter $\epsilon$, i.e.,

$$\sigma_v^2 = S^2\epsilon^2 \left\{ 1 + 3\epsilon^2 < (\kappa^{(2)})^2 > \right\} \left\{ < (\kappa^{(1)})^2 > + \epsilon < \kappa (\kappa^{(2)})^2 > + \epsilon^2 < (\kappa \kappa^{(2)})^{(1)} > \right\}.$$ 

- Plots of the normalized quantities $\sigma_v/\sigma_{sf}$ and $\sigma_v/\sigma^a_v$ versus the vacuum phase advance $\sigma_v$ for periodic step-function lattice with filling factor $\delta = 1/2$. 

![Graph showing normalized quantities vs. $\sigma_v$](image-url)
Evaluation of self-field potential

- Expanding correct to second order in the small parameter $\epsilon$, we obtain the expression for the average potential

$$< \Psi(Q) > = (1 + \epsilon^2 < v^2 >) \phi_0 + \epsilon^2 < v^2 > \left( \phi_1 - \left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] \phi_2 \right.$$

$$+ \frac{1}{2} \left[ \eta^\alpha \eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta} \right] \phi_0 \right),$$

where $v = \kappa^{(2)}$ and the functions $\phi_0(Q)$, $\phi_1(Q)$ and $\phi_2(Q)$ satisfy the Poisson-type equations

$$\nabla_\perp^2 \phi_0 = -s_b n(Q),$$

$$\nabla_\perp^2 \phi_1 = -s_b \left( \left[ Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] + \frac{1}{2} \left[ \eta^\alpha \eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta} \right] \right) n(Q),$$

$$\nabla_\perp^2 \phi_2 = -s_b \left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] n(Q),$$

- In cylindrical coordinates $Q_1 = r \cos(\theta), Q_2 = r \sin(\theta)$

$$\left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] = Q^1 \frac{\partial}{\partial Q^1} - Q^2 \frac{\partial}{\partial Q^2} = \cos(2\theta) r \frac{\partial}{\partial r} - \sin(2\theta) \frac{\partial}{\partial \theta}.$$
Boundary conditions for self-field potential

- Needs to specify some boundary surface in the coordinate space \((Q^1, Q^2)\) and certain boundary conditions on this boundary.

- It is convenient to designate this boundary surface to be a surface in the coordinate space \((Q^1, Q^2)\), where the function \(L(Q, \bar{Q})\) satisfies the same boundary conditions as the function \(L(x, \bar{x})\) in the coordinate space \((x^1, x^2)\).

- In that case, the boundary conditions for \(\phi_0(Q), \phi_1(Q)\) and \(\phi_2(Q)\) in the coordinate space \((Q^1, Q^2)\) are the same as the boundary conditions for the Green’s function \(L(Q, \bar{Q})\).

- Note that this boundary surface in the coordinate space \((Q^1, Q^2)\) becomes a surface that oscillates around the boundary surface in the coordinate space \((x^1, x^2)\).

- Because the two surfaces differ, the average potential \(\bar{\Psi}(Q)\) in the coordinate space \((Q^1, Q^2)\) does not satisfy the same boundary conditions as the un-averaged potential in the coordinate space \((x^1, x^2)\).
Equilibrium self-field potential

• In equilibrium

\[ n(Q) = \int dPG(K) = \int dPG(K_{kin} + K_{ext} + \Phi_0 + \epsilon^2 < v^2 > \bar{\Psi}_1). \]

• For perfectly conducting cylindrical boundary of radius \( R_w \),

\[ \bar{\Psi}_1 = p(r) + \cos(4\theta)q(r), \]

where

\[
\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + sbn'_0(r) \right) p(r) = -sb \frac{8}{R_w^4} \int_0^{R_w} drr^3n_0(r),
\]

\[
\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{16}{r^2} + sbn'_0(r) \right) q(r) = -2sb \left( n_0 + \frac{4}{r^2} \int_0^r d\bar{r}n_0(\bar{r}) - \frac{12}{r^4} \int_0^r d\bar{r}^3n_0(\bar{r}) \right).
\]

• with boundary conditions

\[ p(R_w) = -\frac{2sb}{R_w^2} \int_0^{R_w} drr^3n_0(r), \quad q(R_w) = -sb \left( \frac{2}{R_w^2} \int_0^{R_w} d\bar{r}n_0(\bar{r}) - \frac{1}{2} \int_0^{R_w} d\bar{r}^3n_0(\bar{r}) \right). \]

• Here, \( n_0(r) \equiv \int dPG[K_{kin} + K_{ext} + \Phi_0(r)] \) and \( n'_0 \equiv \partial n_0(r)/\partial \Phi_0 \) with \( \Phi_0(r) \) determined self-consistently through

\[
\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \Phi_0 = -sb n_0(r),
\]
Thermal equilibrium distribution

- Zero-order thermal equilibrium average beam profile

\[ \bar{n}_0 = \frac{n_0(r)}{n_0(0)} = \exp \left( -\frac{< (\kappa^{(1)})^2 > r^2 / 2 + \Phi_0(r)}{\bar{T}} \right). \]

- Normalizing distance as \( \rho^2 = r^2 s_b \bar{n} / T \) and potential as \( \bar{\phi}_0 = \Phi_0 / T, \bar{q} = q / T \).

- For thermal equilibrium distribution normalized quantities are function of two parameters

\[ \epsilon \sim (\sigma_v / 2\pi) \]

and beam intensity

\[ \bar{s} = s_b \bar{n} / (2 < (\kappa^{(1)})^2 >) = \omega_p^2 / 2\omega_{sf}^2 < 1, \]

where \( \omega_p^2 = 4\pi q^2 n_0(0) / m_b \) is the plasma density at the beam center and \( \omega_{sf} = (\sigma_v / 2\pi)(v_b / S) \) is the average focusing frequency of the lattice.
Maxwellian distribution averaged density profile

- Plots of density profiles \( \bar{n}_0(\rho) \) for different values of intensity parameter \( \bar{s} \approx 0.1; 0.3; 0.5; 0.8; 0.9; 0.99; 0.999 \).

- For \( \bar{s} \ll 1 \)

\[
\bar{n}_0(\rho) \approx \left( 1 + \frac{\rho^2}{4\bar{s}} \right)^{\bar{s}} \exp \left( -\frac{\rho^2}{4\bar{s}} \right).
\]

- For \( \Delta = 1/\bar{s} - 1 \ll 1 \)

\[
\bar{n}_0(\rho) \approx \frac{\left[ 1 + \frac{1}{2} \Delta + \frac{1}{24} \Delta^2 \right]^2}{\left[ 1 + \frac{1}{2} \Delta I_0(\rho) + \frac{1}{24} [\Delta I_0(\rho)]^2 \right]^2},
\]

where \( I_0(\rho) \) is zero order modified Bessel function.
Corrections to density profile for Maxwellian distribution

- Normalized equilibrium density profile, accurate to second order in $\epsilon$, can be expressed as

$$\bar{n}(\rho, \theta) = \bar{n}_0(\rho)[1 - \epsilon^2 < \kappa^{(2)} > \cos(4\theta)\bar{q}(\rho)].$$

- Define RMS radius as

$$R^2(\theta) = \frac{\int \rho^3\bar{n}(\rho, \theta)d\rho}{\int \rho\bar{n}(\rho, \theta)d\rho}, \quad R_0^2 = \frac{\int \rho^3\bar{n}_0(\rho)d\rho}{\int \rho\bar{n}_0(\rho)d\rho}.$$

- Relative change in RMS radius as $\delta R(\theta)/R_0 = R(\theta)/R_0 - 1$ can be expressed as

$$\frac{\delta R(\theta)}{R_0} = -\epsilon^2 < v^2 > \cos(4\theta)A(\bar{s}),$$

where $A(\bar{s})$ is given by

$$A(\bar{s}) = \frac{1}{2} \left[ \frac{\int_0^\infty \rho^3\bar{q}(\rho)\bar{n}_0(\rho)d\rho}{\int_0^\infty \rho^3\bar{n}_0(\rho)d\rho} - \frac{\int_0^\infty \rho\bar{q}(\rho)\bar{n}_0(\rho)d\rho}{\int_0^\infty \rho\bar{n}_0(\rho)d\rho} \right].$$
Corrections to density profile, cont’d

• Plots of density profile correction $\bar{q}(\rho)\bar{n}_0(\rho)$ for different values of intensity parameter $\bar{s} = 0.3; 0.6; 0.9; 0.99; 0.999$.

• Plots of function $A(\bar{s})$, (a) linear $s$-scale and (b) logarithmic $s$-scale.

• Here $\rho_{1/2}(\bar{s}) \simeq \ln \left[ \frac{C}{\Delta} \sqrt{2\pi \frac{C}{\Delta}} \right]$ is the beam edge radius $\bar{n}_0(\rho_{1/2}) = 1/2$ and $C \approx 0.78$. 
Specific example of the beam inside of perfectly conducting cylindrical boundary

- Kapchinskij-Vladimirskij distribution $G(K) = (\bar{n}_0/2\pi)\delta(K - K_0)$ with
  $$n_0(r) = \begin{cases} \bar{n}_0, & r < a, \\ 0, & a < r \leq R_w. \end{cases}$$

- Beam radius $r_b$ becomes weakly dependent on the angle $\theta$ according to
  $$r_b(\theta) = a \left( 1 + \epsilon^2 <v^2> \frac{\bar{s}}{1 - \bar{s}} \left( \frac{(a/R_w)^4}{1 - \frac{2\bar{s}}{1 - \bar{s}} \ln \frac{a}{R_w}} + \frac{\cos(4\theta)}{2} \frac{(a/R_w)^8}{1 + \frac{\bar{s}}{4(1 - \bar{s})} \left( 1 - \left( \frac{a}{R_w} \right)^8 \right)} \right) \right)$$

- For $R_w \to \infty$, $\bar{\Psi}_1 = 0$, and the total self-field potential inside the beam is given by zero order potential $\Phi_0$, which is what one would expect for a Kapchinskij-Vladimirskij distribution in free space ($R_w \to \infty$) which generates the constant beam density and linear self-field forces.

- For finite $R_w$, the image charge oscillations produce additional contributions to the average self-field potential inside the beam, which lead to the octopole correction to the average beam radius.
Simulation studies

- The simulations results for a sinusoidal lattice $\kappa(s) = \kappa_0 \sin(2\pi s/S)$

- Beam intensity $\bar{s}_b = 0.5$, wall radius $R_w = 5R_0$, and $\sigma_v = 25^0$ which corresponds to $\epsilon = 0.07$.

- Fourier spectrum $\sim \exp(im\theta)$ of the averaged over time relative RMS radius change $\langle \delta R(\theta)/R_0 \rangle$ as a function of azimuthal mode number $m$.

- Time history of $m = 4$ component of $\delta R(\theta, s)/R_0$. 

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Results and Conclusions

• New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

• Using the particle’s vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order $\epsilon^3$.

• The higher order corrections allow us to extend the average formulaic results to larger vacuum phase advances approaching $\sigma_v \sim 90^0$ with accuracies to within several percent.

• The equations determining the average self-field potential have been derived for general boundary conditions by taking into account the average contribution of the charges induced on the boundary.

• The average equation can be used to find an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

• The time-dependent formulation can be used to describe collective beam dynamics which is slow in the transformed coordinates (slow compared to the period of the lattice).
Results and Conclusions, cont’d

- The corrections to the self-field are small, and therefore the "smooth-focusing" approximation for the self-field potential can be a good approximation even for moderate values of the vacuum phase advance.

- For example, for vacuum phase advance of $\sigma_v = 90^0$ the correction to the RMS radius of the beam described by a thermal equilibrium distribution arising from the corrections to the average self-field potential is of order 0.5%.

- Nonetheless, note that because the average self-field potential acquires an octupole component, the average motion of some beam particles becomes non-integrable and the trajectories become chaotic.

- This chaotic behavior of some of the beam particles may change the nature of the Landau damping (or growth) of collective excitations supported by the beam.

- Also, due to the presence of the extra non-axisymmetric terms in the equations for the self-field potential, the stability properties of different beam quasi-equilibria can change significantly.