Phase Space Manipulation in High-Brightness Electron Beams

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Lawrence Berkeley National Laboratory Seminar
NGLS talk
June-27-2011
Outline

- Introduction/Motivation.
- The Argonne Wakefield Accelerator.
- “Multi-beam” control of electron beam.
- Phase space exchange between two degrees of freedom.
- Development of a single-shot longitudinal phase space diagnostics.
- Production of a train of picosecond relativistic electron bunches.
- Future plans.
Introduction

- Particle accelerators produce and accelerate charged-particles beams up to relativistic energies.

- Accelerators applications include
  - Material sciences (electron microscopy and X-ray in accelerator-based light sources),
  - Medical application,
  - Nuclear and high-energy physics.
Beam & Phase Space: definitions

- A particle is identified by its coordinate and momentum in a 6D phase.

\[ P_i = \{x_i, p_{xi}; y_i, p_{yi}; z_i, p_{zi}\} \]

- A beam is a collection of particles confined in space \( p_z \gg p_x, p_y \)

- Separate to 2D sub-phase space

Transverse space \( \{x_i, p_{xi}\} \) \( \{y_i, p_{yi}\} \) Longitudinal space \( \{z_i, p_{zi}\} \)

- Trace space coordinates:

\[ \vec{X}_i = (x_i, x_i', y_i, y_i', z_i, \delta_i) \]

with \( (x', y') = \frac{p_{(x,y)}}{p_z} \) and \( \delta = \frac{p_z - p_{z,\text{REF}}}{p_z} \)

- Trace space coordinates of a particle downstream of an element can be obtained via

\[ \vec{X}_f = R \vec{X}_i \]

R: transfer matrix of the element
Statistical representation of a beam

- A beam can be represented by its second-order moments arranged as a covariance matrix or “beam matrix”

\[
X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \quad \Sigma = \langle XX' \rangle = \begin{pmatrix}
\langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle & \langle xz \rangle & \langle x\delta \rangle \\
\langle x'x \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle & \langle x'z \rangle & \langle x'\delta \rangle \\
\langle yx \rangle & \langle yx' \rangle & \langle y^2 \rangle & \langle yy' \rangle & \langle yz \rangle & \langle y\delta \rangle \\
\langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'^2 \rangle & \langle y'z \rangle & \langle y'\delta \rangle \\
\langle zx \rangle & \langle zx' \rangle & \langle zy \rangle & \langle zy' \rangle & \langle z^2 \rangle & \langle z\delta \rangle \\
\langle z'x \rangle & \langle z'x' \rangle & \langle z'y \rangle & \langle z'y' \rangle & \langle z'z \rangle & \langle z'\delta \rangle \\
\end{pmatrix}
\]

- Uncoupled 2D phase spaces \(\Rightarrow\) beam matrix is block diagonal. \[\Sigma = \begin{bmatrix} A & B \\ \ast & C \end{bmatrix}\]

- The beam matrix can be propagated using the transfer matrix formalism

\[
\Sigma_f = R \sum_i R^T
\]
**Emittance and Brightness: figure of merit of a beam**

- **Canonical emittance:**
  \[
  \varepsilon_x = \frac{1}{m_e c} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2}
  \]

- **Trace-space emittance** (experimentally measurable)
  \[
  \tilde{\varepsilon}_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}
  \]

- **Normalized Brightness**
  \[
  B = \frac{Q}{\Gamma} = \frac{Q}{\varepsilon_x \varepsilon_y \varepsilon_z}
  \]
  Beam charge

- **Beam’s moment used to parametrize the beam**

\[
\beta x'^2 + \gamma x^2 + 2\alpha xx' = \varepsilon_x
\]

- **Courant-Snyder parameters**
  \[
  \beta = \frac{\langle x^2 \rangle}{\varepsilon_x}, \alpha = -\frac{\langle xx' \rangle}{\varepsilon_x}, \gamma = \frac{\langle x'^2 \rangle}{\varepsilon_x}
  \]
Goals of research work

- Explore phase space manipulations.
- Multi-beam control of the transverse beam parameters.
- Investigate phase space exchange between two degrees of freedom.
- Develop a single shot longitudinal phase space diagnostics and produce a train of picoseconds electron bunches.
**Importance of phase space manipulation: next generation e+/e- linear collider**

- International Linear Collider requirement \( \Rightarrow (\varepsilon_x, \varepsilon_y, \varepsilon_z) = (8, 0.02, 3000) \ \mu m \)

\[
L = \frac{f_R N_+ N_-}{4\pi\varepsilon \sqrt{\beta_x \beta_y}}
\]

\( f_R \) is the repetition frequency. \( \beta_x \) and \( \beta_y \) are the twiss parameters. Assume \( \varepsilon = \varepsilon_x = \varepsilon_z \)

- An RF gun at Q=3.2nC gives \( (\varepsilon_x, \varepsilon_y, \varepsilon_z) = (6,6,13) \ \mu m \)
  \[
  \Rightarrow \Gamma = 480 \mu m^3
  \]

- Redistributing the beam emittances within the 3 degrees of freedom \( \Rightarrow \) suppression of the damping ring (a 3 km circumference ring!)

Importance of phase space manipulation: reducing the size of accelerator-based light sources

- Compact (5 GeV) short-wavelength ($\lambda=1$ Å), x-ray free-electron lasers require
  \[ \epsilon_{x,y} \leq \frac{1}{4\pi} \gamma \lambda \]
  or \( (\epsilon_x, \epsilon_y, \epsilon_z) = (0.1, 0.1, 10) \ \mu m \)

- An RF gun at Q=1 nC gives \( (\epsilon_x, \epsilon_y, \epsilon_z) = (1,1,0.1) \ \mu m \)
  \[ \Rightarrow \Gamma = 0.1 \mu m^3 \]

- Only x-ray FEL (LCLS at SLAC) so far operates at 25 GeV
Source of high-quality electron beams: the photoinjectors

Principle of operation:
- 1+1/2 cell cavity resonating on TM_{010,π} mode
- Laser illuminate photocathode on back plate
- Laser synchronized with e.m. field

Capabilities
- e- beam is naturally bunch,
- e- bunch shape controlled by laser parameters,
- emittances, charge, size are variable
Beam dynamics simulations using Particle-in-Cell codes

- Beam is represented by ensemble of macroparticles.
  \[
  \frac{dP}{dt} = F_{\text{ext}} + F_{\text{sc}} \]
  
  - To compute space charge force \((F_{\text{sc}})\) we use the quasi-static approach.
    1. Lorentz transformation to rest frame
    2. Deposit the charge on 2D or 3D grid
    3. Solve Poisson equation \(\Rightarrow\) electric field.
    4. Inverse Lorentz transformation to Laboratory frame \(\Rightarrow\) B and E fields.
    5. Interpolate E and B field for each of the macro particle position

- ASTRA for 2D cylindrically symmetrical beam low number of macroparticles (between 2000 and 5000).
- IMPACT-T: a fully 3D tracking code, can be run on cluster computers allowing a large number of macroparticles (~ 200,000).
An example of high-brightness photoinjector
The Argonne Wakefield Accelerator (AWA)

- Support advanced accelerator science experiments
- Availability to external user (e.g. NIU)
- Chosen for its versatility
- Overview
  - 5-8 MeV rf gun
  - Linac with 8 MV accelerating voltage
  - Extensive diagnostics

Gun + solenoids
linac
solenoid
spectrometer

constructed as part of my PhD work
Simulation of AWA nominal setup

\begin{align*}
Astra & \text{ (blue)} \ VS \ Impact-T \ (\text{red}) \\
Q & = 1 nC
\end{align*}
Generic beam diagnostics at AWA

- Transverse beam density monitor

- Integrating Current Monitor: Measure beam charge

- Virtual Cathode: Get laser distribution on the photocathode
Generic beam diagnostics at AWA (cont)

- Quadruple Scan Measure emittance
  - Vary quadrupole
  - Measure spot size downstream

\[ \sigma_f^2 = \frac{\varepsilon}{\beta_x} \left[ R_{11}^2(k) - 2R_{11}(k)R_{12}(k)\alpha_x + R_{12}^2(k)\gamma_x \right] \]

- Simulated measurements retrieved 22.75/26.37 vs 23.18/25.55 \( \mu \text{m} \)

- Spectrometer: Measure beam energy

\[ y = \eta \frac{\delta p}{p} \]
\[ \eta = 18.4 \text{cm} \]
“Multi-beam” control of electron beam

- Experiment reveals some interesting physics.
- Interaction of multiple beams can be used to shape/control the parameters of a “main” beam.

- Multibeams also provide intricate distribution for precisely benchmarking multi-particle simulation algorithms.

Potential Applications
- Beam focusing.
- Multi-beam-based manipulation of a beam
- Mimicking and optimizing field-array emitter patterns.

Recent example:
- Halo removal at Tevatron,
- Electron lens at Tevatron.
How to generate a multi-beam electron bunch in a photoinjector?

- mask in the laser path $\Rightarrow$ generation of a multibeamlet distribution
Comparison simulation/experiments

Increasing B-field

$Q=1nC$
Insights from simulations

- Lorentz force integrated over the longitudinal bunch distribution along beamline.
- Most of beam-beam interaction occurs within 5 cm from the cathode surface.
Emittance Exchange Concept

Initial state

\[ \sigma_0 = \begin{pmatrix} \varepsilon_{x0}T_{x0} & 0 \\ 0 & \varepsilon_{z0}T_{z0} \end{pmatrix} \]

\[ T_{u0} = \begin{pmatrix} \beta_{u0} & -\alpha_{u0} \\ -\alpha_{u0} & \gamma_{u0} \end{pmatrix} \]

\[ \varepsilon_x^2 = \text{det}(\sigma_{xx}) = \langle x^2 \rangle \langle x'^2 \rangle - \langle x'x \rangle^2 \]

\[ \varepsilon_z^2 = \text{det}(\sigma_{zz}) = \langle z^2 \rangle \langle \delta^2 \rangle - \langle z\delta \rangle^2 \]

\[ M_{EX} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \]

Final State

\[ \sigma_f = M_{EX}\sigma_0M_{EX}^T \]

\[ \sigma_f = \begin{pmatrix} \varepsilon_{z0}T_{z0}BB^T & 0 \\ 0 & \varepsilon_{x0}T_{x0}CC^T \end{pmatrix} \]

- Need \( M_{EX} \) \( \Rightarrow \) \( \varepsilon_{xf} = \varepsilon_{z0} \) & \( \varepsilon_{zf} = \varepsilon_{x0} \)

- Coordinates swap between transverse and longitudinal spaces.

- From now on, we use 4D notations
Deflector Cavity Design and modeling

- Key element in phase space exchange

Field in a pillbox cylindrical cavity at zero-crossing

\[ E_z = E_0 k x e^{-i\omega t} \approx E_0 k x \]
\[ c B_y = i E_0 e^{-i\omega t} \approx E_0 k z \]
\[ \delta = \kappa x \quad \Delta x' = \kappa z \]

- Cavity normalized strength

\[ \kappa = \frac{2\pi e V_0}{\lambda E} \]
Phase space exchange theory

\[ M_{DL} = \begin{bmatrix} 1 & L & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \xi \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Total transport matrix of the exchanger is

\[ \kappa = -\frac{1}{\eta} \]

\[ \begin{bmatrix} 1 + \eta \kappa & 2L(1 + \eta \kappa) & L\kappa & 2\eta + LR_{56}\kappa + \eta^2 \kappa \\ 0 & 1 + \eta \kappa & \kappa & R_{56}\kappa \\ R_{56}\kappa & 2\eta + LR_{56}\kappa + \eta^2 \kappa & 1 + \eta \kappa & 2R_{56}(1 + \eta \kappa) \\ \kappa & L\kappa & 0 & 1 + \eta \kappa \end{bmatrix} \]

\[ M_{cav} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \kappa & 0 \\ 0 & 0 & 1 & 0 \\ \kappa & 0 & 0 & 1 \end{bmatrix} \]
Limitations for exact emittance exchange

- Exchanger matrix:

\[ M = \begin{bmatrix}
0 & \frac{23\lambda}{128} & -\frac{128L + 64L_c - 23\lambda}{128\eta} & \eta - \frac{R_{56}(128L + 64L_c - 23\lambda)}{128\eta} \\
0 & 0 & -\frac{23\lambda}{128\eta} & \frac{23R_{56}\lambda}{128\eta^2} \\
-\frac{R_{56}}{\eta} & \eta + \frac{R_{56}(23\lambda - L - L_c)}{128\eta} & \frac{23R_{56}^2\lambda}{128\eta^2} & \frac{23R_{56}^2\lambda}{128\eta^2} \\
\eta & -1 & -\frac{23\lambda}{128\eta} & \frac{23R_{56}\lambda}{128\eta^2}
\end{bmatrix} \]

- Emittance not perfectly exchanged

\[ \varepsilon_x^2 = \varepsilon_{x0}^2 + \Lambda_x^2 \varepsilon_{x0} \varepsilon_{z0} \]
\[ \varepsilon_z^2 = \varepsilon_{x0}^2 + \Lambda_z^2 \varepsilon_{x0} \varepsilon_{z0} \]

\[ \Lambda^2 = \frac{529\lambda_c^2(1 + \alpha_x^2)}{16384\eta^2 \beta_x \beta_z} \left( R_{56}^2 \left(1 + \alpha_z^2\right) - 2R_{56} \alpha_z \beta_z + \beta_z^2 \right) \]

- Coupling Terms, can minimized with respect to chirp:

\[ \Lambda^2 = \frac{529\lambda_c^2 R_{56}^2 \left(1 + \alpha_x^2\right)}{16384\eta^2 \beta_x \beta_z} \text{ for } \alpha_z = -\frac{\langle z\delta \rangle}{\varepsilon_z} \]

Deflecting Cavity wavelength

- We need a quadrupole magnets upstream of the exchanger
Limitations for exact emittance exchange

- Real particle distribution with incoming emittance \((e_x, e_z) = (15.9,3.75)\) mm

\[
\Delta_x = \left( \frac{E_x}{E_{x0}} \right) - 1
\]
\[
\Delta_z = \left( \frac{E_z}{E_{z0}} \right) - 1
\]

No space charge

Choose this region

Q = 100pC

- Space charge does not prevent the minimization of emittance dilution
Investigation of emittance exchange via start-to-end simulation of AWA

- Cathode to exchanger entrance modeled with ASTRA output passed to IMPACT-T for simulation of exchanger beamline

- Optimized C-S parameters (space charge on)
  \[ \alpha_x = 10.2; \beta_x = 13.54 \text{ m} \]

- Summary of emittance dilutions

<table>
<thead>
<tr>
<th>Space Charge</th>
<th>( \varepsilon_{z4}(\mu m) )</th>
<th>( \varepsilon_{xf}(\mu m) )</th>
<th>( \varepsilon_{zt}(\mu m) )</th>
<th>( \varepsilon_{zf}(\mu m) )</th>
<th>( \Delta_x(%) )</th>
<th>( \Delta_z(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>OFF</td>
<td>22.30</td>
<td>2.90</td>
<td>4.4</td>
<td>22.67</td>
<td>51%</td>
<td>16%</td>
</tr>
<tr>
<td>ON</td>
<td>21.58</td>
<td>2.54</td>
<td>4.7</td>
<td>20.90</td>
<td>85%</td>
<td>-4%</td>
</tr>
</tbody>
</table>
**Measured initial emittance partition**

<table>
<thead>
<tr>
<th>Symbol (unit)</th>
<th>ASTRA</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$ (pC)</td>
<td>100</td>
<td>100 ± 10</td>
</tr>
<tr>
<td>laser $\sigma_t$ (ps)</td>
<td>1.95</td>
<td>1.85 ± 0.2</td>
</tr>
<tr>
<td>rms laser size (mm)</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>gun field (MV/m)</td>
<td>43.92</td>
<td>47 ± 2</td>
</tr>
<tr>
<td>gun phase (deg.)</td>
<td>65</td>
<td>60 ± 5</td>
</tr>
<tr>
<td>booster field (MV/m)</td>
<td>15.75</td>
<td>15.5 ± 1</td>
</tr>
<tr>
<td>booster phase (deg.)</td>
<td>50.35</td>
<td>52 ± 4</td>
</tr>
<tr>
<td>L1 peak B-field (T)</td>
<td>0.062</td>
<td>0.0618 ± 0.0031</td>
</tr>
<tr>
<td>L2 peak B-field (T)</td>
<td>-0.062</td>
<td>-0.0626 ± 0.003</td>
</tr>
<tr>
<td>L3 peak B-field (T)</td>
<td>-0.228</td>
<td>-0.228 ± 0.0114</td>
</tr>
<tr>
<td>$\varepsilon_x$ ($\mu$m)</td>
<td>19.5</td>
<td>18.5 ± 2</td>
</tr>
<tr>
<td>$\varepsilon_y$ ($\mu$m)</td>
<td>19.5</td>
<td>16.2 ± 2</td>
</tr>
<tr>
<td>$\varepsilon_z$ ($\mu$m)</td>
<td>7.40</td>
<td>-</td>
</tr>
</tbody>
</table>

Longitudinal emittance is inferred from the energy spread measurement~8mm

Transverse emittance measured using Quadrupole scan technique
Phase space exchange: experimental plans

- AWA can achieved an interesting emittance partition $\varepsilon_z < \varepsilon_x$

- Next step was to design and construct a phase space exchange beamline
  - Not possible due to space and time constraints.
  - Construct simpler beamline to commission the hardware especially the deflecting cavity
  - **Configure the beamline for other purpose: a single-shot longitudinal phase space diagnostics**
Single-shot longitudinal phase space measurement

- Map initial \((z, d)\) longitudinal phase space to the transverse plane \((x, y)\)

![Diagram of single-shot longitudinal phase space measurement with relevant components labeled: Dipole, QE1, QE2', QE3, Deflecting Cavity, and YE6. The diagram includes arrows indicating the flow of the measurement process and phase space plots for visualization.]
Theoretical background

Typically $\Delta E/E = \text{a few mrad} \Rightarrow \Delta x = \text{a few mm's}$

$\Delta z = R_{56} \Delta E/E$

$\Delta z \equiv (z'_2 - z'_1) - (z_2 - z_1)$

To preserve the relative distance between particles $\Rightarrow \Delta z = 0 \Rightarrow R_{56} = 0 \Rightarrow$

**Goal to map longitudinal phase space to screen**

$$x = \eta \delta_o + h_x$$

$$y = \kappa z_0 + h_y + R_{56} \delta_0 \kappa$$

**Higher order terms**

Determine the resolution

$$\mathbf{F}$$

$E_z = E_0 \frac{y}{a} \cos \omega t$

$B_x = \frac{E_0}{\omega a} \sin \omega t \Rightarrow F_y \neq 0 \Rightarrow \Delta y = \kappa \cdot \Delta z$

$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2}$

$H_{xy} = h_x \frac{\gamma}{\gamma^2} \left(x^2\right) + h_y \left(y^2\right) - 2h_x h_y \langle xy \rangle$
Commissioning of the deflecting cavity

- Developed beam-based calibration procedure to determine cavity deflecting strength

Vertical displacement on screen versus phase of TDC

\[ \delta y = 13.75 \sin(\varphi - 153.1) + 0.47 \]

Calibration procedure for TDC strength

\[ \kappa = 1.68 m^{-1} \]

\[ \delta z = \Delta \varphi \left( \frac{230}{360} \right) \]
Commissioning of the deflecting cavity (cont)

- Measured deflecting $k$ as a function of input power is in good agreement with numerical simulations.
- The cavity was operated up to 800 kW but conditioned to its nominal 2.3 MW power without problem.

\[ \text{Simulation scaling} \]

![Graph showing $k$ as a function of power](image)

- Screen
- Deflecting cavity
Dispersion measurements

- Beam Based measurement of dispersion is used in order to indirectly tune the R56

Dispersion measurement at YE6 for different QE1 strength

Dispersion versus QE1 strength

QE1 = 0.0 T/m

QE1 = 0.2 T/m

QE1 = 0.3 T/m

QE1 = 0.4 T/m

QE1 = 1.4 T/m gives R56 = 0
Single shot measurement of the LPS

Using calibration procedure, we can convert the configuration space coordinates into longitudinal coordinate and fractional momentum spread.

Q = 1.5 nC
E = 14.6 MeV

\[ \eta = 0.4 m \]
\[ \kappa = 1.7 m^{-1} \]
**Generation of train of bunches**

- Generate bunch with tunable spacing. 4 pulses generated using a-BBO crystal.
Generation of train of bunches measurement

- Evolution of the longitudinal phase space associated to a train of four bunches as a function of the quadrupole QE1.

![Graphs showing evolution of longitudinal phase space for different values of QE1](image-url)
Generation of train of bunches: applications

- Resonant excitation wakefield in dielectric-loaded waveguides
- Production of narrow-band radiation in the Terahertz (THz) regime

**z-spacing vs. quadrupole strength**

[Graph showing z-spacing vs. quadrupole strength]

**Modulated distribution and corresponding spectrum**

[Graphs showing modulated distribution and corresponding spectrum]
Summary of achievement and future plans

- Advanced beam controls in a photoinjector:
  - Developed and tested a technique to use a multi-beam arrangement to control the beam properties via “multi-beam” interaction.

- Emittance Exchange:
  - Designed a emittance exchanger beamline and explore limiting effects,
  - Installed and commissioned key components of the exchanger
  - Verified initial emittance partitions of AWA

- Longitudinal phase space diagnostics:
  - Designed, build a single-shot longitudinal phase space diagnostics
  - Use the beamline to produce a train of ps electron bunches

- Future Plans:
  - Developed longitudinal phase space diagnostics to
    - Explore velocity bunching in photoinjector
    - Beam dynamic in beam-driven wakefield accelerators
  - Designed exchanger beamline will be installed at AWA
    - Current shaping for enhancing performance of beam-drive wakefield acceleration
Thank you
List of publications published or submitted


- M. Rihaoui, P. Piot, J.G. Power, W. Gai, “Verification of the AWA photoinjector beam parameters required for a transverse-to-longitudinal emittance exchange experiment”. In the Proceeding of Particle Accelerator Conference (PAC’09), Vancouver, Canada (May 2009)


List of publications published or submitted


List of publications published or submitted

- M. Rihaoui, C. L. Bohn, P. Piot and J. G. Power, “Impact of transverse irregularities at the photo- cathode on the production of high-charge electron bunches”. In the Proceedings of Particle Accelerator Conference (PAC 07), Albuquerque, New Mexico, 25-29 Jun 2007, pp 4027


Backup slides
Magnets modeling

**Magnetic Quadrupoles:**
Perform measurements of B field for the AWA quads

**Magnetic dipole:**
- Magnetic field profile and magnets are from RadiaBe
- Ideal magnetic dipole have hard edge model. We model the magnetic dipoles with fringe fields using Enge Coefficients

\[
\frac{B_y}{B_{y0}} = \frac{1}{1 + \exp(c_i s^{i-1})}, \quad i = 1, \ldots, 8
\]

\[
s = \frac{z - z_0}{g}
\]

Enge Coefficients fit
Transfer matrix of a realistic system

• Use a realistic model to test for the exchanger validation.
• Generate Initial particle distribution of 6 particles with offset in position and momentum with a reference particle \( X = 0 \).
• to get the six phase space \( R \) transfer matrix

\[
X = 0 \quad \text{Reference particle}
\]

\[
X_i = \alpha_i \, e_i \quad \text{Probe particles}
\]

\[
X \xrightarrow{R} Y = RX \quad \text{Ref}
\]

\[
X_i \xrightarrow{R} Y_i = RX_i \quad \text{Probe}
\]

\[
\delta_i Y_i \equiv Y_i - Y = R[X_i - X]
\]

\[
\begin{bmatrix}
R_{1i} \\
\vdots \\
R_{6i}
\end{bmatrix}
= \frac{\delta_i Y}{\alpha_i}
\]
RF deflecting Cavity

\[ d_1 = 25 \text{cm} \quad \text{and} \quad d_2 = 25 \text{cm} \]

\[ M_{\text{cav}} = \begin{bmatrix}
    1 & 0.7294 & 1.4262 & 0 \\
    0 & 0.9974 & 3.909 & 0 \\
    0 & 0 & 0.9956 & 0 \\
    3.91 & 1.426 & 0.732 & 1.003
\end{bmatrix} \]

\[ L_c = d_1 + \lambda + d_2 \]

\[ M_{\text{theory}} = M_{d_2} M_3 M_2 M_1 M_{d_1} = \begin{bmatrix}
    1 & L_c & \frac{\kappa L_c}{2} & 0 \\
    0 & 1 & \kappa & 0 \\
    0 & 0 & 1 & 0 \\
    \kappa & \frac{L_c}{2} & \frac{23}{128} (\kappa^2 \lambda) & 1
\end{bmatrix} = \begin{bmatrix}
    1 & 0.73 & 1.427 & 0 \\
    0 & 1 & 3.909 & 0 \\
    0 & 0 & 1 & 0 \\
    3.909 & 1.427 & 0.631 & 1
\end{bmatrix} \]

From Don Edwards notes*

The transfer matrix for one cell deflecting cavity using pillbox model is:

\[ M_i = \begin{bmatrix}
    1 & L_i & \frac{\kappa L_i}{2} & 0 \\
    0 & 1 & \kappa & 0 \\
    0 & \kappa L_i & \frac{\kappa^2 L_i}{4} & 1
\end{bmatrix} \]

*Note on rf deflecting cavity can be found at:
http://www.nicadd.niu.edu/aard/emittance_exchange/
Transfer matrix of a realistic emittance-exchanger beamline

**Matrix inferred from particle tracking**

\[
M_{DL-CAV-DL} = \begin{bmatrix}
-0.0010 & 0.0858 & 8.4 & -0.266 \\
-0.0015 & 0.015 & 3.896 & 0.2355 \\
0.2387 & 0.2471 & 0.0233 & 0.0022 \\
3.896 & 8.409 & 0.745 & 0.0436
\end{bmatrix}
\]

**Matrix analytically derived and evaluated for** \( \kappa = \frac{-1}{\eta} \)

\[
M = \begin{bmatrix}
0 & \frac{23\lambda}{128} & -\frac{128L + 64L_c - 23\lambda}{128\eta} & \eta - \frac{R_{56}(128L + 64L_c - 23\lambda)}{128\eta} \\
0 & 0 & \frac{-1}{\eta} & \frac{R_{56}}{\eta} \\
\frac{-R_{56}}{\eta} & \frac{R_{56}}{\eta} & \frac{23R_{56}\lambda}{128\eta^2} & \frac{23R_{56}^2\lambda}{128\eta^2} \\
\frac{-1}{\eta} & \frac{128L + 64L_c - 23\lambda}{128\eta} & \frac{23\lambda}{128\eta^2} & \frac{23R_{56}\lambda}{128\eta^2}
\end{bmatrix} = \begin{bmatrix}
0 & 0.041 & 8.3 & 0.2456 \\
0.236 & 0.2456 & 0.038 & 0.002 \\
3.909 & 8.3 & 0.631 & 0.038
\end{bmatrix}
\]

**Realistic model reproduce the matrix analytically derived using hard-edge elements**
Simulations Tools cont...

POISSON used to generate B field

Photo cathode (B = 0)

SUPERFISH used to generate E field

Magnetic field in the solenoid

bucking  focusing  matching

Electric field in the rf gun π mode