Variation on the Theme of Madey’s Theorem

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FEL Oscillator using a high Q cavity is a low gain device

- Low gain oscillator output has characteristics complementary to the high-gain single pass FEL
  - Stability, coherence, high rep rate, gentler pulses
The complement of XFELs will not be complete without an x-ray FEL oscillator

- Bragg crystal as high reflectivity mirror was suggested at 1983 BNL WS
- Highly stable, fully coherent, ps pulses with ultra-high spectral purity (10^-7) with MHz rep rate—“a real laser”
- XFELO output pulses are copies of the same circulating intra-cavity pulse → possibility to produce x-ray spectral combs, allowing experimental x-ray quantum optics for fundamental physics
Madey’s two theorems (1979) are of fundamental importance in a low gain FEL

- **First Theorem:**
  \[
  \langle (\Delta \gamma)^2 \rangle = \frac{2\pi^2}{m^2 c \omega^2} E_0^2 \left. \frac{d^3 w_s}{d^2 \phi_\perp d\omega} \right|_{\phi_\perp=0}
  \]

- **Second Theorem:**
  \[
  \langle \Delta \gamma \rangle = \frac{1}{2} \left. \frac{\partial}{\partial \gamma_0} \langle (\Delta \gamma)^2 \rangle \right|_{\phi_\perp=0}
  \]

- **Gain:**
  \[
  g = - \frac{j}{q} \left. \frac{(2\pi)^3}{m \omega^2} \frac{\partial}{\partial \gamma} \frac{d^3 w_s}{d \omega d^2 \phi_\perp} \right|_{\phi_\perp=0}
  \]

- **Amplitude of coherent radiation:**
  \[
  \frac{E_0}{d^2 \phi_\perp d\omega}
  \]

- **Angular-spectral density of spontaneous emission**
  \[
  \frac{d^3 w_s}{d^2 \phi_\perp d\omega}
  \]
Variation on the theme of Madey’s theorem

- FEL gain from quantum mechanical view
- Derivation of Madey theorems
  - Mainly from electron dynamics (the most familiar)
  - Mainly from radiation process (the most intriguing)
  - Maxwell-klimontovich equations, to include 3D effects
Low gain FEL from quantum mechanics (J.M.J. Madey, 1971)

- Transition amplitudes for emission and absorption

\[ [a, a^+] = 1 \rightarrow \langle n+1|a^+|n \rangle = \sqrt{n+1}; \quad \langle n-1|a|n \rangle = \sqrt{n} \]

\[ \langle n+1; p'_e | (aJ^+ + a^+J) | n; p_e \rangle = \langle n+1|a^+|n \rangle \langle p'_e |J| p_e \rangle = \sqrt{n+1}A_e \]

\[ \langle n-1; p'_a | (aJ^+ + a^+J) | n; p_a \rangle = \sqrt{n}A_a; \quad A_a = \langle p'_a | J^+ | p_a \rangle \]
Electron-photon interaction in the presence of external potential

- **2-momenta:**
  \[ p = \left( E, E - \frac{m^2}{2E} \right), \quad k = (\omega, \omega), \quad q = (0, -Q), \quad \bar{q} = (0, Q) \]

- **Conservation:**
  \[ p_e + \hbar q = \hbar k + p'_e, \quad p_a + \hbar \bar{q} = \hbar k + p'_a \]

- **Emission case:**
  \[ E_e = E'_e + \hbar \omega, \quad E_e - \frac{m^2}{2E_e} - Q = E'_e - \frac{m^2}{2E'_e} + \hbar \omega \]

  \[ \therefore \omega = \frac{2E_e^2}{m^2} Q \left[ 1 + \frac{\hbar \omega}{E_e} \right]; \quad Q = \frac{k_U}{(1 + K^2 / 2)} \]

- **Quantum corrections to classical resonance condition:**
  \[ E_e \approx E_c + \hbar \omega_c / 2; \quad E_a \approx E_c - \hbar \omega_c / 2 \]
Gain

The net energy production from the emission and absorption processes

\[
\frac{dW}{d\omega} = N_e \hbar \omega \left[ (n + 1) \left( \Gamma_e \right) - n \Gamma_a \right] = N_e \hbar \omega \left[ n \left( \Gamma_e - \Gamma_a \right) + \Gamma_e \right]
\]

\[
\Gamma_e = \Gamma(E_c + \hbar \omega_c / 2), \quad \Gamma_a = \Gamma(E_c - \hbar \omega_c / 2)
\]

\[
\frac{dW}{d\omega} = N_e n(\hbar \omega)^2 \frac{d}{dE_c} \Gamma
\]

The classical gain formula when \( \hbar \omega = E \)

\[
g = -\frac{j \left( 2\pi \right)^3}{q m \omega^2} \frac{\partial}{\partial \gamma} \frac{d^3 \omega_s}{d \omega d^2 \phi_{\perp}} \bigg|_{\phi_{\perp}=0}
\]
**Derivation mainly from electron dynamics**

*(Madey, 1979; W. Colson, 1977)*

- **H** (the electron motion in a static radiation device) \(\gg \Delta H\) (EM field-electron interaction)

- Solve to second order in \(\Delta H\) compute electron energy change \(<\Delta \gamma>\) and the spread \(<\Delta \gamma^2>\)
  - Explicitly solve the pendulum equation if undulator or do the Hamiltonian perturbation theory for a general radiation device

→ The second theorem

→ The first theorem by invoking from conservation of electron and EM energy

- Since Madey 1979, the second theorem has been proven to increasing generality by N. Kroll, S. Krinsky-J.M. Wang-P. Luchini, and V.N. Litvinenko-N.A. Vinokurov
Derivation mainly from EM field point of view (Max Zolotorev, 1995?)

- Electrons with energy $mc^2\gamma_j$ interacting with the laser field $E_0$ changing its energy and producing spontaneous emission field $\varepsilon$

- Make us of three basic principles:
  - EM field amplitudes add linearly
  - EM energy is quadratic in field amplitude
  - Conservation of total energy (EM field and electrons)
**Derivation (KJK after Max)**

- **Incident laser field (assumed to be plane wave):**
  \[
  E(x_\perp, z_I, t) = E_L(z_I, t) = E_0 \cos \omega(t - z_I / c)
  \]

- **Outgoing laser + spontaneous emission:**
  \[
  E(x_\perp, z_{II}, t) = E_0 \cos \omega(t - z_{II} / c) + \sum_j \varepsilon(x_\perp, z_{II}, t - t_j; \gamma_j)
  \]

- **Energy associated with the spontaneous emission:**
  \[
  \bar{\gamma}_j = \gamma_j + \xi \Delta \gamma_j; \quad \Delta \gamma_j = \gamma'_j - \gamma_j; \quad 0 \leq \xi \leq 1
  \]

- **\(\Delta \gamma\) is small:**
  \[
  \varepsilon(x_\perp, z_{II}, t - t_j; \gamma_j) = \varepsilon(x_\perp, z_{II}, t - t_j; \gamma_j) + \xi \Delta \gamma_j \frac{\partial}{\partial \gamma_j} \varepsilon(x_\perp, z_{II}, t - t_j; \gamma_j)
  \]

- **Energy conservation**
  \[
  mc^2 \sum_j \Delta \gamma_j = -\frac{c}{4\pi} \int dx_\perp dt \left( E^2(x_\perp, z_{II}, t) - E^2(x_\perp, z_I, t) \right)
  \]
**Derivation (ctn’d)**

- **Energy conservation:**

  \[
  \Delta \gamma_j = - \frac{E_0}{2\pi mc} \int dx_\perp dt \cos \omega \left( t - z_{II} / c \right) \times \\
  \left( \varepsilon \left( x_\perp, z_{II}, t - t_j; \gamma_j \right) + \xi \sum_j \Delta \gamma_j \frac{\partial}{\partial \gamma_j} \varepsilon \left( x_\perp, z_{II}, t - t_j; \gamma_j \right) \right)
  \]

- **Iterate once:**

  \[
  \Delta \gamma_j = - \frac{E_0}{2\pi mc} \int dx_\perp dt \cos \omega \left( t - z_{II} / c \right) \varepsilon \left( x_\perp, z_{II}, t - t_j; \gamma_j \right) + \\
  \left( \frac{E_0}{2\pi mc} \right)^2 \xi \int dx_\perp ' dt ' \cos \omega \left( t' - z_{II} / c \right) \varepsilon \left( x_\perp ', z_{II}, t' - t_j; \gamma_j \right) \times \\
  \frac{\partial}{\partial \gamma_j} \int dx_\perp dt \cos \omega \left( t - z_{II} / c \right) \varepsilon \left( x_\perp, z_{II}, t - t_j; \gamma_j \right)
  \]
**Spontaneous emission field**

- **Paraxial, far-field representation:**
  \[ \varepsilon(x_\perp, z, t; \gamma) = \frac{1}{(2\pi)^{3/2}} \int d^2\phi_\perp d\omega e^{ik(\phi_\perp g x_\perp + \sqrt{1-\phi_\perp^2} z) - i\omega t} \mathcal{E}(\phi_\perp, \omega; \gamma) \]

- **Angular-spectral density of emitted energy**
  \[ \frac{d^3w_s}{d^2\phi_\perp d\omega} = \frac{c}{2\pi k^2} |\mathcal{E}(\phi_\perp, \omega)|^2 \]

- **The interference term (laser-spontaneous emission)**
  \[ \int dx_\perp dt \cos \omega(t - z_{II}/c) \varepsilon(x_\perp, z_{II}, t - t_j; \gamma_j) \]
  \[ = \frac{(2\pi)^{3/2}}{2k^2} \left( e^{i\omega t_j} \mathcal{E}_0(\omega, \gamma_j) + e^{-i\omega t_j} \mathcal{E}_0^*(\omega, \gamma_j) \right) \]

- **Here**
  \[ \mathcal{E}_0(\omega, \gamma) = \mathcal{E}(\phi_\perp = 0, \omega; \gamma) \]
Music finally!

- The iteration equation again:

\[
\Delta \gamma_j = -\frac{E_0 \sqrt{\pi}}{mk^2 \sqrt{2}} \left( e^{i\omega_j} \vartheta_0(\omega, \gamma_j) + c.c \right) +
\]

\[
\xi \left( \frac{E_0 \sqrt{\pi}}{mk^2 \sqrt{2}} \right)^2 \left( e^{i\omega_j} \vartheta_0(\omega, \gamma_j) + c.c \right) \frac{\partial}{\partial \gamma_j} \left( e^{-i\omega_j} \vartheta_0(\omega, \gamma_j) + c.c \right)
\]

- Assume random phase:

\[
\frac{1}{N_e} \sum_j e^{i\omega_j} = 0, \quad \frac{1}{N_e} \sum_{j,k} e^{i\omega(t_j-t_k)} = 1
\]

- \( \langle \Delta \gamma \rangle \) from the second term and \( \langle \Delta \gamma^2 \rangle \) from the square of the first term

- Obtain the second theorem

- Obtain the first theorem if \( \xi = 1 \)

The electron energy associated with the spontaneous emission appears to be the final energy!!
Remarks

Assumptions in this derivation:
- The spontaneous power is negligible compared to the laser power
- The length of the undulator $\ll$ Synchrotron oscillation period of the pendulum motion

Taking into account the electron’s energy change on the radiation process is to take into account the radiation reaction—the run-away problem in radiation reaction is avoided by looking at far field radiation only
Taking 3-D effects into account

- 3D-effects: transversely finite electron beam and radiation field
- Start from the coupled Maxwell-Klimontovich equation derived originally for high-gain analysis
- The equations can be solved straightforwardly in perturbation theory by using “integration over unperturbed trajectory” technique
- The third order term represents the EM field with gain
- In general, it is necessary to diagonalize the complex gain matrix (P. Luchini and S. Solimeno, 1986)
- However,..
3D Madey’s theorem in the absence of transverse focusing (generally a good approximation for a low gain system)

- Introduce a certain “undulator field” $U(\phi; \eta)$
- Form the brightness functions $B_U(x, \phi, \eta)$ and $B_A(x, \phi)$ as the Wigner distribution of the undulator and laser field A.
- 3D Madey’s theorem (F: electron distribution):

$$G = \frac{n_e \chi_1 \chi_2}{\lambda^2} \frac{\int d\eta d\phi dx dy \ B_E(y, \phi) B_U(\eta, x - y, \phi - p) \frac{\partial}{\partial \eta} F(\eta, x, p)}{\int d\phi dy \ B_E(y, \phi)},$$

- Gaussian

$$G = \frac{I}{I_A} \frac{4 \sqrt{2\pi^2} K^2 [JJ]^2 N_u^3 \lambda_1^{3/2} \lambda_u^{1/2}}{(1 + K^2/2)^{3/2}} \left[ 1 + \sum_{z} \left( z - s \right) \left\{ \sin[2x(z - s)] - i \cos[2x(z - s)] \right\} \right] \left[ k_1 L_u \sum_\phi^2 + \frac{L_u}{4k_1 \Sigma_z^2} \right]$$
Gain calculation for XFELO

Figure 5.8: (a) Gain curves as a function of the frequency detuning $x = \pi N_u \Delta \nu$ for (1) $\epsilon_x = \lambda_1/4\pi$, $\sigma_\eta = 1/6N_u$; (2) $\epsilon_x = \lambda_1/4\pi$, $\sigma_\eta = 1/3N_u$; and (3) $\epsilon_x = \lambda_1/2\pi$, $\sigma_\eta = 1/3N_u$. The radiation Rayleigh range $Z_R$ and beam focusing $Z_\beta$ have been chosen to maximize $G$, which has been normalized to the maximum gain of (1). (b) Normalized gain, maximized over $\Delta \nu$, plotted as a function of $Z_\beta/L_u$ and $Z_R/L_u$ for $\epsilon_x = \lambda_1/4\pi$ and $\sigma_\eta = 1/2N_u$. 
Madey’s two theorems have both fundamental as well as practical interest

- Illustrate general principles in action, such as energy conservation, EM linearity, Hamiltonian mechanics,..
- Gain calculation of exotic system such as XU
- Gain enhancement with optical klystron or similar schemes invented to narrow the spectrum of high-gain FELs such as iSASE or pSASE

Could there be a high-gain version of Madey’s theorem?

- 1D M-K equations together with the quasi-linear theory (Z. Huang,..)

\[
\langle \hat{\eta}^2 \rangle - \langle \hat{\eta} \rangle^2 \approx 2P_{FEL}(z) / \rho P_{beam}; \quad \hat{\eta} = \delta \gamma / (\gamma \rho)
\]

Variation on Madey Theorems