Variation on the Theme of Madey's Theorem

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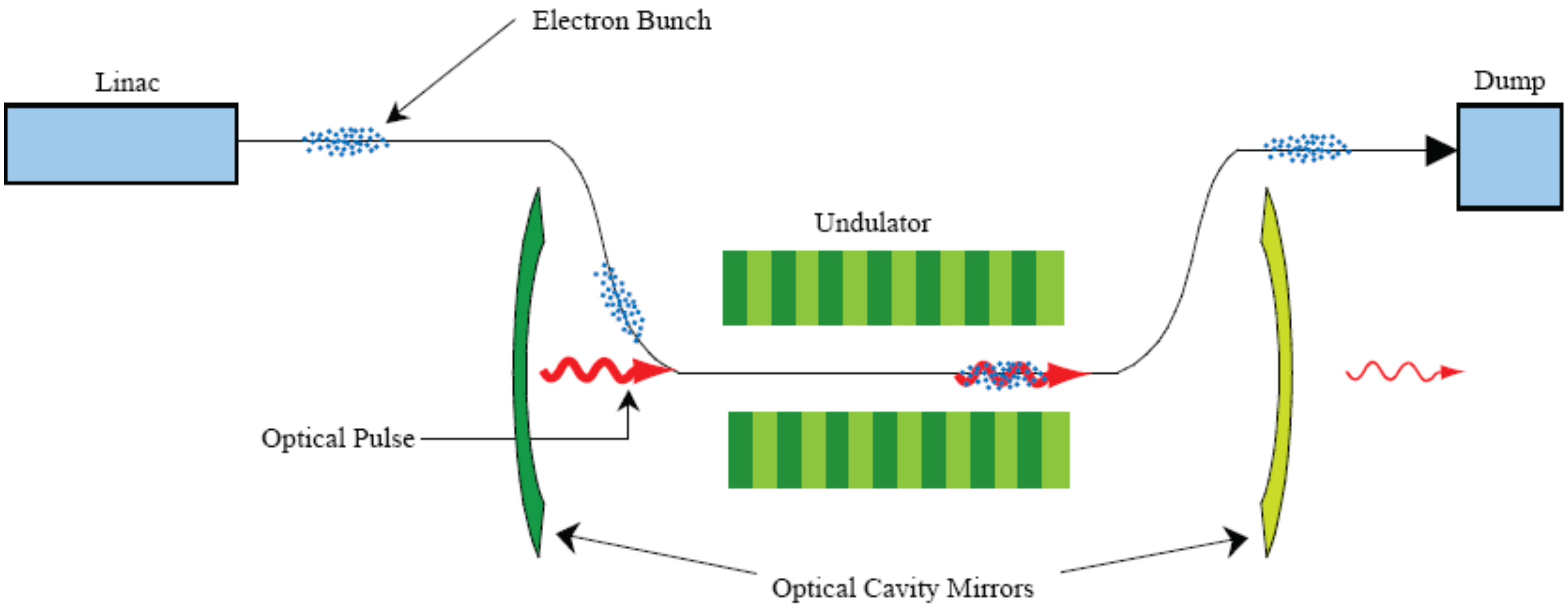
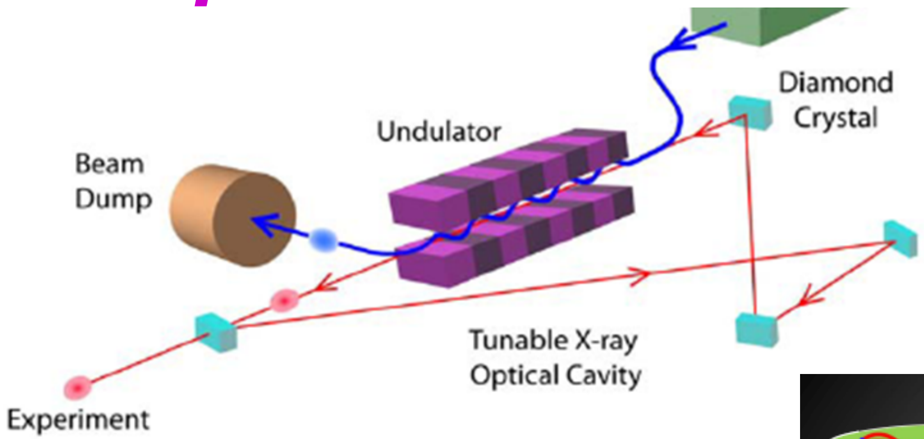
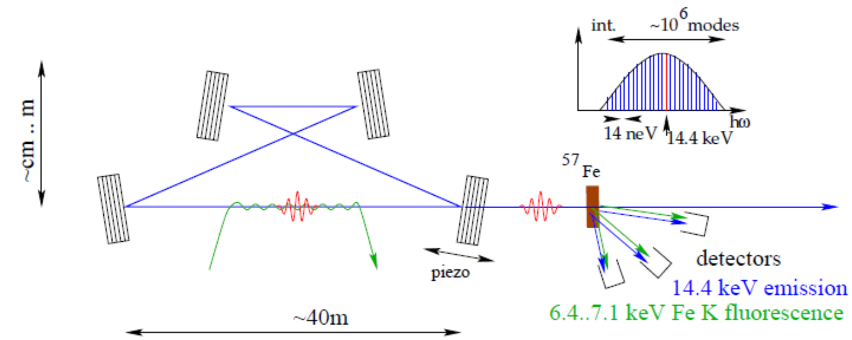
FEL Oscillator using a high Q cavity is a low gain device



- **Low gain oscillator out put has characteristics complementary to the high-gain single pass FEL**
  - **Stability, *coherence, high rep rate, gentler pulses***

The complement of XFELs will not be complete without an x-ray FEL oscillator





- **Bragg crystal as high reflectivity mirror was suggested at 1983 BNL WS**
- **Highly stable, fully coherent, ps pulses with ultra-high spectral purity (10^{-7}) with MHz rep rate— "*a real laser*"**
- **XFELO output pulses are copies of the same circulating intra-cavity pulse → possibility to produce x-ray spectral combs, allowing experimental x-ray quantum optics for fundamental physics**

Madey's two theorems (1979) are of fundamental importance in a low gain FEL

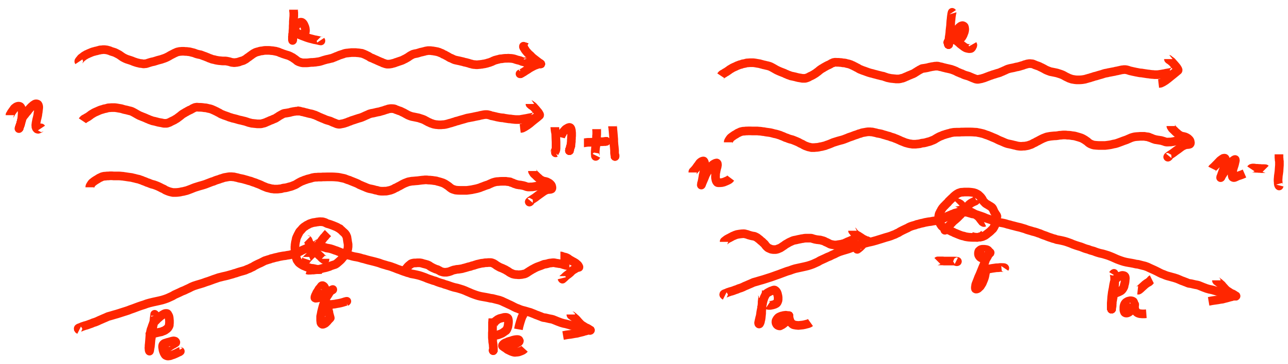
- **First Theorem:** $$\left\langle (\Delta\gamma)^2 \right\rangle = \frac{2\pi^2}{m^2 c\omega^2} E_0^{\,2} \left.\frac{d^3 w_S}{d^2\phi_\perp d\omega}\right|_{\phi_\perp = 0}$$
- **Second Theorem:** $$\langle \Delta\gamma \rangle = \frac{1}{2}\frac{\partial}{\partial\gamma_0}\left\langle (\Delta\gamma)^2 \right\rangle$$
- **Gain:** $$g = -\frac{j}{q}\frac{(2\pi)^3}{m\omega^2}\frac{\partial}{\partial\gamma}\left.\frac{d^3 w_s}{d\omega d^2\phi_\perp}\right|_{\phi_\perp = 0}$$

- ***Amplitude of coherent radiation:*** E_0
- ***Angular-spectral density of spontaneous emission*** $$\frac{d^3 w_S}{d^2\phi_\perp d\omega}$$

Variation on the theme of Madey's theorem

- **FEL gain from quantum mechanical view**
- **Derivation of Madey theorems**
  - **Mainly from electron dynamics (the most familiar)**
  - **Mainly from radiation process (the most intriguing)**
  - **Maxwell-klimontovich equations, to include 3D effects**

Low gain FEL from quantum mechanics (J.M.J. Madey, 1971)



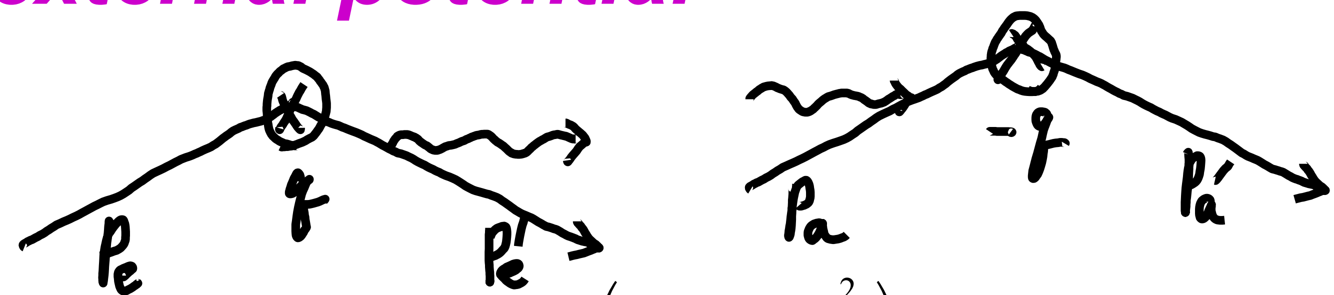
- **Transition amplitudes for emission and absorption**

$$\left[a,a^{+}\right]=1\rightarrow\left\langle n+1\middle|a^{+}\middle|n\right\rangle=\sqrt{n+1};\quad\left\langle n-1\middle|a\middle|n\right\rangle=\sqrt{n}$$

$$\left\langle n+1;p_e'\middle|\left(aJ^{+}+a^{+}J\right)\middle|n;p_e\right\rangle=\left\langle n+1\middle|a^{+}\middle|n\right\rangle\left\langle p_e'\middle|J\middle|p_e\right\rangle=\sqrt{n+1}A_e$$

$$\left\langle n-1;p_a'\middle|\left(aJ^{+}+a^{+}J\right)\middle|n;p_a\right\rangle=\sqrt{n}A_a;\quad A_a=\left\langle p_a'\middle|J^{+}\middle|p_a\right\rangle$$

Electron-photon interaction in the presence of external potential



- **2-momenta;** $p = \left(E, E - \dfrac{m^2}{2E} \right), k = (\omega, \omega), q = (0, -Q), \bar{q} = (0, Q)$
- **Conservation:** $p_e + \hbar q = \hbar k + p_e' \, , \quad p_a + \hbar \bar{q} = \hbar k + p_a'$
- **Emission case:** $E_e = E_e' + \hbar\omega, \quad E_e - \dfrac{m^2}{2E_e} - Q = E_e' - \dfrac{m^2}{2E_e'} + \hbar\omega$

$$\therefore \omega = \frac{2E_e^{\,2}}{m^2} Q \left[1 + \frac{\hbar\omega}{E_e} \right]; \quad Q = \frac{k_U}{(1 + K^2/2)}$$

- **Quantum corrections to classical resonance condition:**

$$E_e \approx E_c + \hbar\omega_c / 2; \quad E_a \approx E_c - \hbar\omega_c / 2$$

Gain

- **The net energy production from the emission and absorption processes**

$$\frac{dW}{d\omega} = N_e \mathrm{h}\omega\left[(n+1)\left(\Gamma_e\right) - n\Gamma_a\right] = N_e \mathrm{h}\omega\left[n\left(\Gamma_e - \Gamma_a\right) + \Gamma_e\right]$$

$$\Gamma_e = \Gamma(E_c + \mathrm{h}\omega_c / 2),\ \Gamma_a = \Gamma(E_c - \mathrm{h}\omega_c / 2)$$

$$\frac{dW}{d\omega} = N_e n(\mathrm{h}\omega)^2 \frac{d}{dE_c}\Gamma$$

→ **The classical gain formula when** $\mathrm{h}\omega = E$

$$g = -\frac{j}{q}\frac{\left(2\pi\right)^3}{m\omega^2}\frac{\partial}{\partial\gamma}\frac{d^3 w_s}{d\omega d^2\phi_\perp}\bigg|_{\phi_\perp = 0}$$

Derivation mainly from electron dynamics (Madey,1979; W. Colson, 1977)

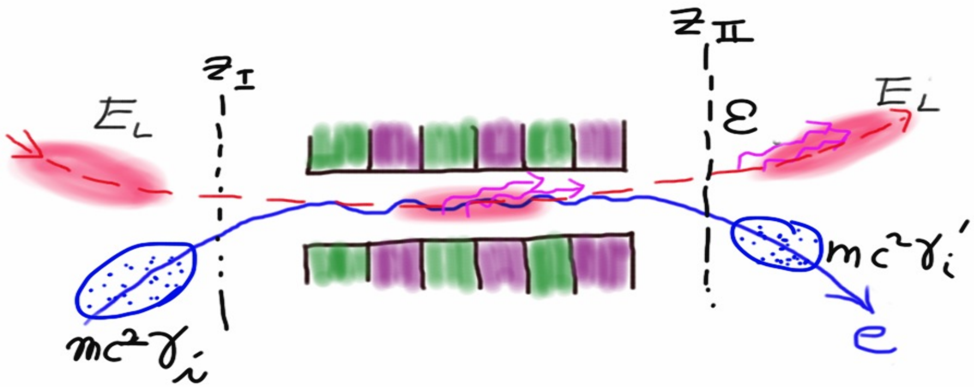
- H (the electron motion in a static radiation device) >>ΔH (EM field-electron interaction)
- Solve to second order in ΔH compute electron energy change $\langle\Delta\gamma\rangle$ and the spread $\langle\Delta\gamma^2\rangle$
  - Explicitly solve the pendulum equation if undulator or do the Hamiltonian perturbation theory for a general radiation device

→ The second theorem

→ The first theorem by invoking from conservation of electron and EM energy

- Since Madey 1979, the second theorem has been proven to increasing generality by N. Kroll, S. Krinsky-J.M. Wang-P. Luchini, and V.N. Litvinenko-N.A. Vinokurov

Derivation mainly from EM field point of view (Max Zolotorev, 1995?)



- **Electrons with energy $mc^2\gamma_j$ interacting with the laser field E_0 changing its energy and producing spontaneous emission field ε**
- **Make us of three basic principles:**
  - **EM field amplitudes add linearly**
  - **EM energy is quadratic in field amplitude**
  - **Conservation of total energy (EM field and electrons)**

Derivation (KJK after Max)

- Incident laser field (assumed to be plane wave):

$$E(\mathbf{x}_\perp, z_I, t) = E_L(z_I, t) = E_0 \cos\omega(t - z_I / c)$$

- Outgoing laser + spontaneous emission:

$$E(\mathbf{x}_\perp, z_{II}, t) = E_0 \cos\omega\left(t - z_{II} / c\right) + \sum_j \varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \overline{\gamma}_j\right)$$

- **Energy associated with the spontaneous emission:**

$$\overline{\gamma}_j = \gamma_j + \xi\Delta\gamma_j; \quad \Delta\gamma_j = \gamma_j' - \gamma_j; \quad 0 \le \xi \le 1$$

- $\Delta\gamma$ is small:

$$\varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \overline{\gamma}_j\right) = \varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right) + \xi\Delta\gamma_j \frac{\partial}{\partial\gamma_j}\varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right)$$

- Energy conservation

$$mc^2 \sum_j \Delta\gamma_j = -\frac{c}{4\pi}\int d\mathbf{x}_\perp dt\left(E^2(\mathbf{x}_\perp, z_{II}, t) - E^2(\mathbf{x}_\perp, z_I, t)\right)$$

Derivation (ctn'd)

- **Energy conservation:**

$$\Delta\gamma_j = -\frac{E_0}{2\pi mc}\int d\mathbf{x}_\perp dt \cos\omega\left(t - z_{II}/c\right)\times$$
$$\left(\varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right) + \xi\sum_j \Delta\gamma_j \frac{\partial}{\partial\gamma_j}\varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right)\right)$$

- **Iterate once:**

$$\Delta\gamma_j = -\frac{E_0}{2\pi mc}\int d\mathbf{x}_\perp dt \cos\omega\left(t - z_{II}/c\right)\varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right) +$$
$$\left(\frac{E_0}{2\pi mc}\right)^2 \xi\int d\mathbf{x}_\perp{}' dt' \cos\omega\left(t' - z_{II}/c\right)\varepsilon\left(\mathbf{x}_\perp{}', z_{II}, t' - t_j; \gamma_j\right)\times$$
$$\frac{\partial}{\partial\gamma_j}\int d\mathbf{x}_\perp dt \cos\omega\left(t - z_{II}/c\right)\varepsilon(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j)$$

Spontaneous emission field

- **Paraxial, far-field representation:**

$$\varepsilon(\mathbf{x}_\perp, z, t; \gamma) = \frac{1}{(2\pi)^{3/2}} \int d^2\phi_\perp d\omega e^{ik(\phi_\perp \cdot \mathbf{x}_\perp + \sqrt{1-\phi^2} z) - i\omega t} \tilde{\mathcal{E}}(\phi_\perp, \omega; \gamma)$$

- **Angular-spectral density of emitted energy**

$$\frac{d^3 w_S}{d^2\phi_\perp d\omega} = \frac{c}{2\pi k^2} \left| \tilde{\mathcal{E}}(\phi_\perp, \omega \right|^2$$

- **The interference term (laser-spontaneous emission)**

$$\int d\mathbf{x}_\perp dt \cos\omega\left(t - z_{II} / c\right) \varepsilon\left(\mathbf{x}_\perp, z_{II}, t - t_j; \gamma_j\right)$$

$$= \frac{(2\pi)^{3/2}}{2k^2} \left(e^{i\omega t_j} \tilde{\mathcal{E}}_0(\omega, \gamma_j) + e^{-i\omega t_j} \tilde{\mathcal{E}}_0^*(\omega, \gamma_j)\right)$$

- **Here** $\tilde{\mathcal{E}}_0(\omega, \gamma) = \tilde{\mathcal{E}}(\phi_\perp = 0, \omega; \gamma)$

Music finally!

- **The iteration equation again:**

$$\Delta\gamma_j = -\frac{E_0\sqrt{\pi}}{mk^2\sqrt{2}}\left(e^{i\omega t_j}\tilde{\varepsilon}_0(\omega,\gamma_j) + c.c\right) +$$

$$\xi\left(\frac{E_0\sqrt{\pi}}{mk^2\sqrt{2}}\right)^2\left(e^{i\omega t_j}\tilde{\varepsilon}_0(\omega,\gamma_j) + c.c\right)\frac{\partial}{\partial\gamma_j}\left(e^{-i\omega t_j}\tilde{\varepsilon}_0(\omega,\gamma_j) + c.c\right)$$

- **Assume random phase:** $\frac{1}{N_e}\sum_j e^{i\omega t_j} = 0, \quad \frac{1}{N_e}\sum_{j,k} e^{i\omega\left(t_j - t_k\right)} = 1$
- **$\langle\Delta\gamma\rangle$ from the second term and $\langle\Delta\gamma^2\rangle$ from the square of the first term**
- **Obtain the second theorem**
- **Obtain the first theorem if** $\xi = 1$

→The electron energy associated with the spontaneous emission appears to be the final energy !!

Remarks

- Assumptions in this dervation:
  - The spontaneous power is negligible compared to the laser power
  - The length of the undulator << Synchrotron oscillation period of the pendulum motion

- **Taking into account the electron's energy change on the radiation process is to take into account the radiation reaction—the run-away problem in radiation reaction is avoided by looking at far field radiation only**

Taking 3-D effects into account

- 3D-effects: transversely finite electron beam and radiation field
- Start from the coupled Maxwell-Klimontovich equation derived originally for high-gain analysis
- The equations can be solved straightforwardly in perturbation theory by using “integration over unperturbed trajectory” technique
- The third order term represents the EM field with gain
- In general, it is necessary to diagonalize the complex gain matrix (P. Luchini and S. Solimeno, 1986)
- However,..

3D Madey's theorem in the absence of transverse focusing (generally a good approximation for a low gain system)

- Introduce a certain "undulator field" $U(\phi;\eta)$
- Form the brightness functions $B_U(x,\phi,\eta)$ and $B_A(x,\phi)$ as the Wigner distribution of the undulator and laser field A.
- 3D Madey's theorem (F: electron distribution):

$$G = \frac{n_e \chi_1 \chi_2}{\lambda^2} \frac{\int d\eta dp d\phi dx dy\; \mathcal{B}_E(\boldsymbol{y},\phi)\mathcal{B}_U(\eta,\boldsymbol{x}-\boldsymbol{y},\phi-\boldsymbol{p})\frac{\partial}{\partial\eta}\bar{F}(\eta,\boldsymbol{x},\boldsymbol{p})}{\int d\phi dy\; \mathcal{B}_E(\boldsymbol{y},\phi)},$$

- Gaussian

$$G = \frac{I}{I_A}\frac{4\sqrt{2}\pi^2K^2[\mathrm{JJ}]^2}{(1+K^2/2)^{3/2}}\frac{N_u^3\lambda_1^{3/2}\lambda_u^{1/2}}{2\pi\Sigma_x^2}\int_{-1/2}^{1/2} ds dz\; e^{-(z-s)^2N_u^2\sigma_\eta^2/2\pi^2}$$

$$\times \frac{(z-s)\left\{\sin[2x(z-s)] - i\cos[2x(z-s)]\right\}}{1+zs\frac{L_u^2\Sigma_\phi^2}{\Sigma_x^2} - i(z-s)\left[k_1L_u\Sigma_\phi^2 + \frac{L_u}{4k_1\Sigma_x^2}\right]}$$

Gain calculation for XFELO

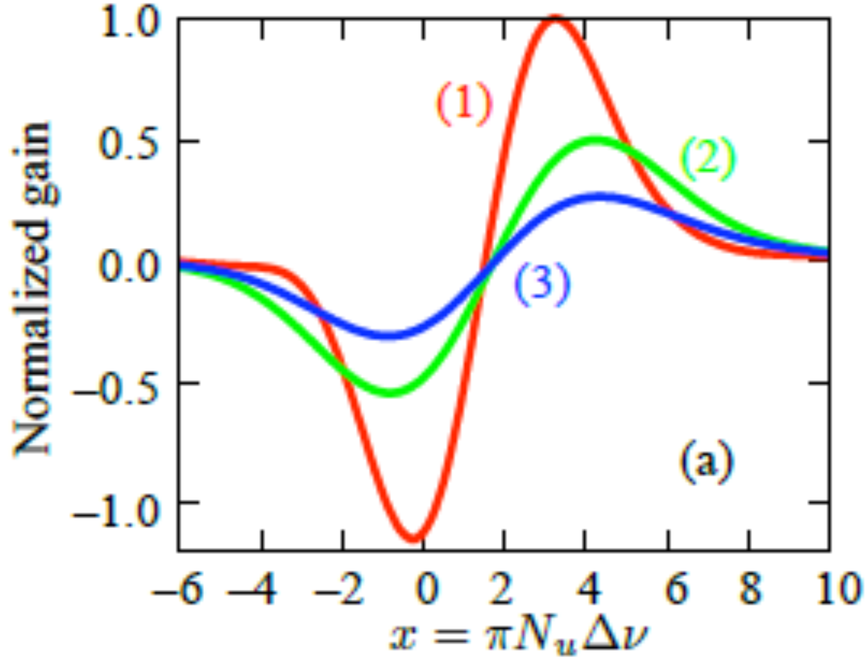


Figure 5.8: (a) Gain curves as a function of the frequency detuning $x = \pi N_u \Delta\nu$ for (1) $\varepsilon_x = \lambda_1/4\pi$, $\sigma_\eta = 1/6N_u$; (2) $\varepsilon_x = \lambda_1/4\pi$, $\sigma_\eta = 1/3N_u$; and (3) $\varepsilon_x = \lambda_1/2\pi$, $\sigma_\eta = 1/3N_u$. The radiation Rayleigh range Z_R and beam focusing Z_β have been chosen to maximize G, which has been normalized to the maximum gain of (1). (b) Normalized gain, maximized over $\Delta\nu$, plotted as a function of Z_β/L_u and Z_R/L_u for $\varepsilon_x = \lambda_1/4\pi$ and $\sigma_\eta = 1/2N_u$.

Madey's two theorems have both fundamental as well as practical interest

- **Illustrate general principles in action, such as energy conservation, EM linearity, Hamiltonian mechanics,..**
- **Gain calculation of exotic system such as XU**
- **Gain enhancement with optical klystron or similar schemes invented to narrow the spectrum of high-gain FELs such as iSASE or pSASE**
- ***Could there be a high-gain version of Madey's theorem?***
  - **1D M-K equations together with the quasi-linear theory (Z. Huang,..)**

$$\left\langle \hat{\eta}^2 \right\rangle - \left\langle \hat{\eta} \right\rangle^2 \approx 2P_{FEL}(z) / \rho P_{beam}; \quad \hat{\eta} = \delta\gamma / (\gamma\rho)$$