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Integrable Systems

For Accelerators

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Talk outline

- Nonlinear systems in general
- Integrable systems – definitions and properties
- Accelerator specifics
- List of available integrable lattices for accelerators
- Big picture, connection to Intensity Frontier

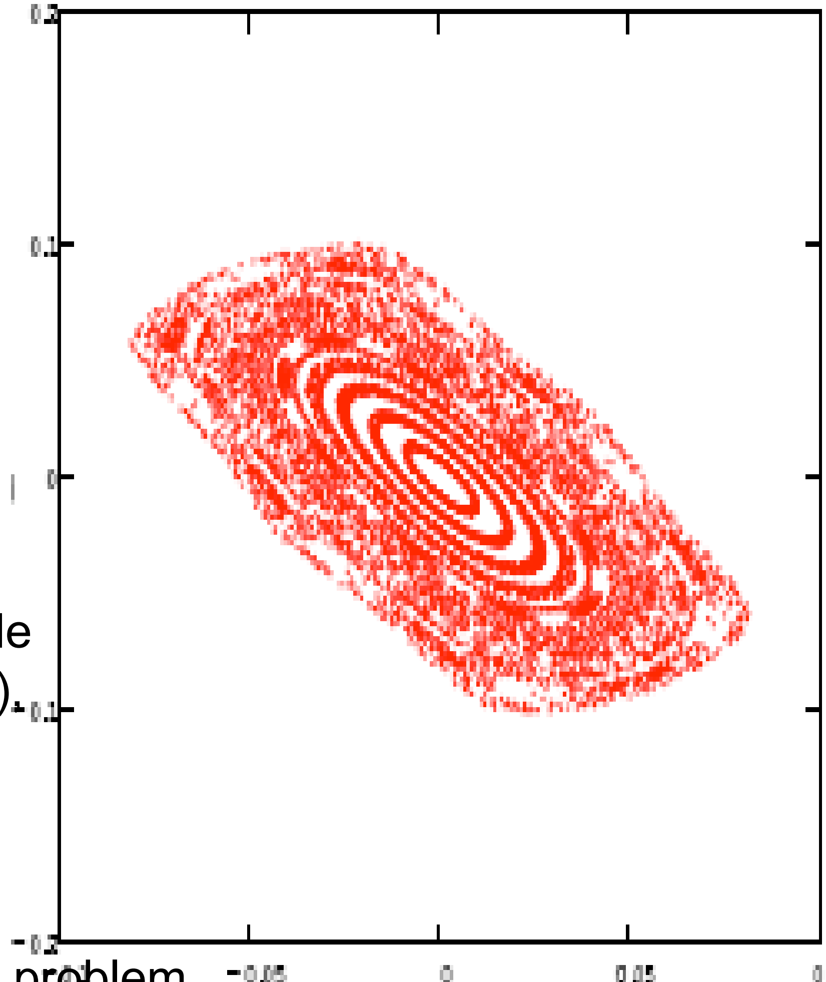
Nonlinear systems in general

1) All solutions of classical mechanics, known by end of XIX century – integrable

2) Nonintegrable systems constitute majority of all real systems
(1st examples, H. Poincare, 1895)

3) In accelerators, any arbitrary nonlinearity (sextupoles, octupoles, etc.) is nonintegrable

4) They are characterized by infinite number of resonances, chaotic motion around unstable points (homoclinic and heteroclinic structures), diffusion, particle loss, and beam-blow-up
(on the right, horizontal phase space of 2D motion in linear lattice with 1 octupole)



5) Even most mathematicians gave up on the problem
V. Arnol'd – topological classification is impossible
V. Lazutkin – it is not known if the area of chaotic motion is finite in general

Integrable systems – exclusive and rare case of nonlinear systems

1) $$E = \frac{m(dx/dt)^2}{2} + U(x) = const$$ $$t = \int \frac{\sqrt{m}dx}{\sqrt{2(const - U(x))}}$$ 1D – case –integrate and inverse the function to get x(t)

2) Integration is possible when invariant is "simple";

3) N-dimensions – N commuting "good" invariants needed for integration (Liouville theorem);

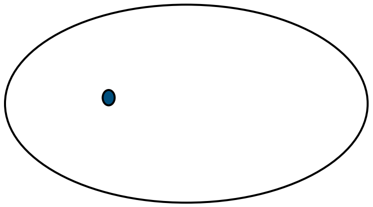
4) All systems have invariants, but they are very complex in general;

5) We need simple invariants for predictable motion.

Perturbed integrable systems? KAM theorem statement – volume of chaotic motion is exponentially small – perturbations<10%?

Main goal – elimination of resonances and introducing large tune spread

Accelerator Specifics



Kepler's problem – integrable
2 Invariants – angular momentum and Hamiltonian (energy)
Accelerator systems have many restrictions as compared to this case

1) **Potential is finite in vacuum chamber**

2) **Hamiltonian is time-dependent and is not invariant any more (accelerators need insertions for injection, extraction, RF, etc.)**

3) **Fields obey Maxwell equations – severe limiting factor in finding integrable potentials**

4) **Need Hamiltonians of** $H = \frac{P_x^2 + P_y^2}{2m} + U(x,y,t)$ **type**

On the way to integrability (resonance suppression)

There are important steps made toward elimination of resonances (integrability)

1) Colliding beams:

a) Round – angular momentum conservation- 1D motion for r (Novosibirsk, 80's, realized at VEPP2000, tune shift around 0.15 achieved);

b) Crab waist - decoupling x and y motion (P. Raimondi (2006), tune shift 0.1 achieved at DAΦNE), factor 2 increase is probable.

2) Numerical methods to eliminate resonances (J. Cary and colleagues, 1994-present);

3) Exact solutions for realization– our goal. The list is presented in next slides

List of 1D Lattices (distributed fields)

1) Linear forces + $\frac{1}{\beta}U(\frac{x}{\sqrt{\beta}})$ (β - beta-function, U -arbitrary function). In normalized variables (NV)

$$d\psi = ds/\beta, X = x/\sqrt{\beta}, X' = \sqrt{\beta}x' - x\beta'/(2\sqrt{\beta})$$

we get time independent Hamiltonian. This transformation - foundation of IOTA lattice, invariant is quadratic in momentum.

2) Any integrable case after NV transformation is again integrable. The other transformation $x = X + D(t), p = P + D'(t)$

So there are classes of integrable systems.

3) Invariants higher order in momentum (Danilov, Perevedentsev (D-P) EPAC 1996);

We have vast variety of integrable systems. The reason – lots of choice *U(x,t)* – 2D function.

McMillan nonlinear optics

- **In 1967 E. McMillan published a paper**

SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

Edwin M. McMillan

September 5, 1967

- **Final report in 1971. This is what later became known as the "McMillan mapping":**

$$x_i = p_{i-1}$$

$$p_i = -x_{i-1} + f(x_i)$$

$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

$$Ax^2p^2 + B(x^2p + xp^2) + C(x^2 + p^2) + Dxp = \text{const}$$

If *A* = *B* = **0** one obtains the Courant-Snyder invariant

1D lattices with thin nonlinear lenses

Variety of solutions shrinks – only 1 and 2 thin lens solutions are known

1) McMillan lattice –1 thin lens;

2) Generalizations (D-P, 1992-1995) – 1 and 2 thin lenses (π/2 phase advance in between):

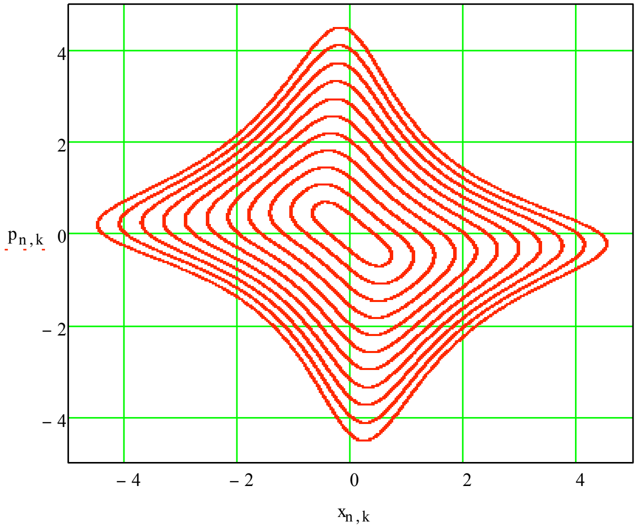
a) $$f_1(f_2)(x) = -\frac{b(c)x^2 + dx + g(h)}{ax^2 + c(b)x + e(f)}$$

b) Combination of logarithmic, polynomial, exponential, or trigonometric functions of coordinate *(*D, PAC 2009*)*. Invariants are trigonometric, exponential, or polynomial in coordinates and momentum.

3) Up to 6th order invariants in momentum (D-P,1995)

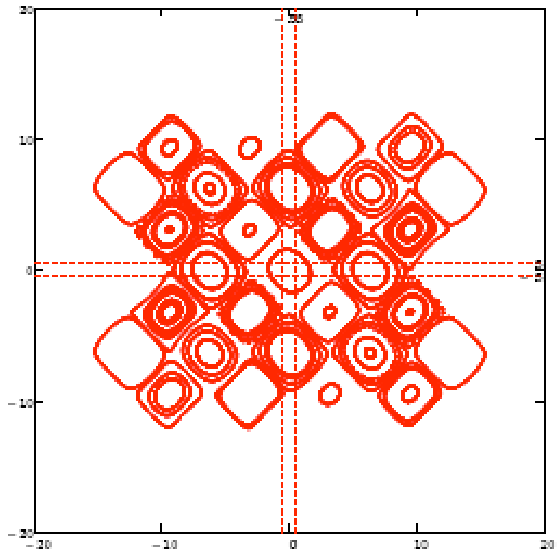
4) More than 2 different lens solutions unknown.

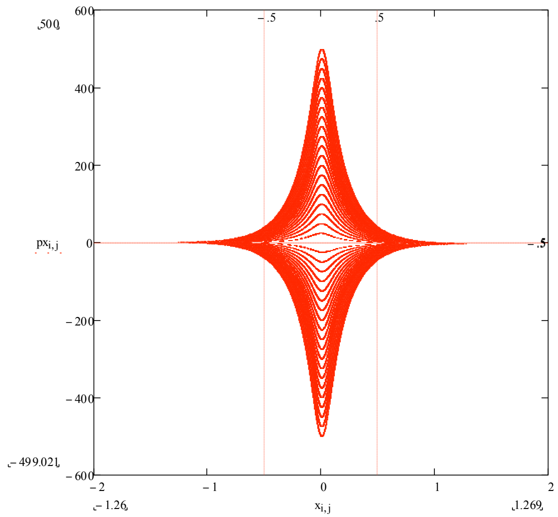
Collection of 1D integrable cases



McMillan symmetric lens case.
As the amplitude grows, the frequency approaches bare lattice tune 1/4

Phase space with undulator-like lens (left), and exponential invariants (right)





$$f(x) = i \ln((\exp(ix) + 5)/(\exp(-ix) + 5))$$

Logarithmic and exponential lenses

2D case with real fields

1) All thin lens solutions can be carried over to 2D case after replacing coordinates and momentum with complex ones $x \to x + iy, p \to p_x - ip_y$ This special form is dictated by Laplace equation for magnetic or electrostatic lenses. All cases give unstable motion due to sum resonance.

2) Round colliding beams – can be realized only with space charge, present in vacuum chamber:

a) 1 or 2 thin lenses with radial kicks $f_1(f_2)(r) = \dfrac{ar}{br^2 + c_1(c_2)}$

b) Time dependent potential $\dfrac{1}{\beta} U(\dfrac{r}{\sqrt{\beta}})$.

3) Approximate cases – J. Cary & colleagues + big decoupling of x-y motion and use of 1D solutions;

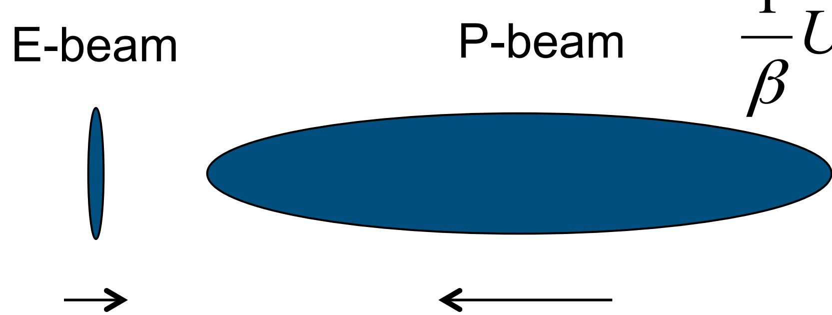
4) Stable integrable motion without space charge in Laplace fields – the only known exact case is IOTA case (Danilov, Nagaitsev, *PRSTAB* 2010); Sergei's talk

Connection to “big picture”- colliders

- Round beams and crab-waist scheme lead to reduction of resonances. Further reduction may result in factor 2 increase in beam-beam tune shift
- Presented two cases of round beam “integrable” kicks can be utilized directly in ERHIC collisions
- It will result in reducing electron beam size and halo for recycling e-beam in energy recovery linac
- All new colliders can in principle incorporate “integrable” colliding beams

$$\frac{1}{\beta}U\left(\frac{r}{\sqrt{\beta}}\right) \textit{ potential}$$

$$f(r) = \frac{ar}{br^2 + c}$$



Connection to big picture - instabilities

- Fast instabilities can be damped sometimes only by betatron frequency spread. Examples:

a) PSR (Los Alamos) – feedback gives 25% threshold increase for e-p instability; octupoles produce large beam loss and can't be used (M. Plum, private communication);

b) SNS Ring – feedback can dump only some modes; some unstable modes of instability appear not at betatron tunes (nu_inst=0.5-2*nu_betatron, probably, some higher modes are excited?);

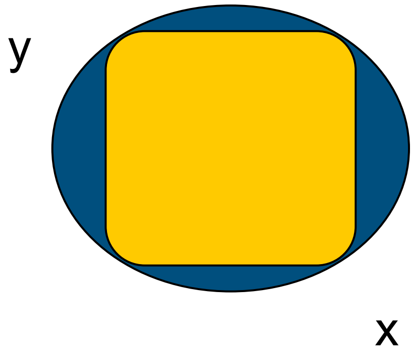
c) Old design of VLHC – resistive wall instability has increments of around 1/3 of revolution time.

Connection to big "picture"- space charge

- There are cases when space charge can be incorporated into integrable lattice (was under development by TECH-X). Best distribution is an open question – space charge limited machines can use these solutions. Examples: boosters, medical accelerators of TRAP type – synchrotron with low injection energy http://www.protominternational.com
- Halo growth in linacs can be mitigated by nonlinear elements (David's talk).
- If space charge is compensated by electron cloud, the current is limited only by e-p instability – this is the best case of applicability of integrable optics since it kills instabilities. Example of application – gamma quants generator, based on 1.7 MeV proton beam and carbon gas (V. Dudnikov, C. Ankenbrandt, IPAC 2011);

Starting point- nonlinear KV

- **The reason for this choice – the first step has to be simple, solvable, and continuously transformable to well-known linear cases for comparisons. Is it really possible to create it self-consistently?**



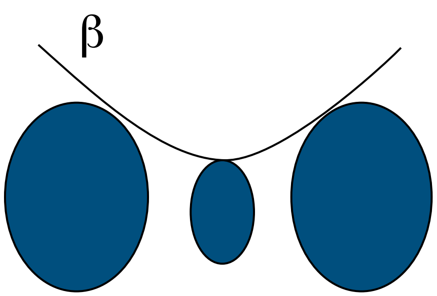
configuration space

1) Let's forget about time dependence for a moment. Nonlinear KV distribution f=C $\delta(H-H_0)$ (H is Hamiltonian in normalized variables) produce constant density but non elliptic shape (can be verified);
2) Yellow figure is what we get for special octupole lattice with one invariant for strong nonlinearity;
3) Let's add external current with the same density (blue color) to form an ellipse;
4) Such combination creates linear fields inside the beam (yellow shape)-same x-y defocusing;
5) The external currents could be moved far away from the beam to create same fields inside it (at least approximately, e.g. making first multipoles in fields expansion);
6) External current has to be proportional to beam current;
7) If they are formed by image current (David's proposal) this happens automatically.

Nonlinear KV- cont.

- **Linear fields from the beam – what is the point?**

They can be integrated into integrable optics!

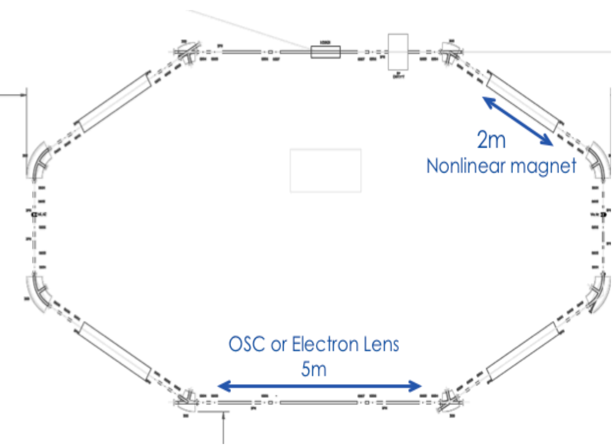


Transverse beam size at three different locations of nonlinear insert

1) Let's recall the time dependence – initial beta function (solid line) depends on time, and the beam size (and its space charge field) depends on longitudinal position as well;
2) One has to recalculate nonlinear elements of the lattice (they depend on beta) using new self-consistent beta function;
3) The external fields to make space charge linear have to vary with longitudinal position accordingly;
4) After all these steps we get nonlinear integrable motion for arbitrary space charge;
5) We get spread of tunes in spite of all particles have one same integral of motion, because the second integral varies in wide range;

Nonlinear KV- cont.

- **More important details**

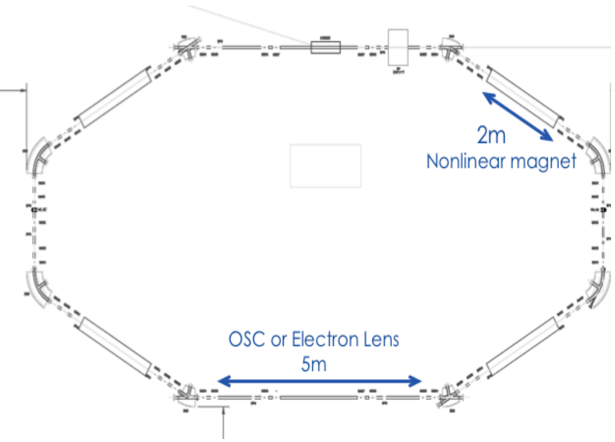


IOTA ring layout

1) This distribution is periodic and the motion is self-consistent but it lacks one property of linear KV- it doesn't preserve its shape if mismatched (David's talk);
2) It needs matching not only to space charge force, but also adjusting nonlinear elements according to beam density;
3) The exact solutions only valid for simple lattice (thin round lens and nonlinear insert), used in David's talk;
4) If the thin lens is realized by several quads and straights, the numerical continuation is possible, but I don't see how we can do it analytically;
5) The conclusion – it is a nice starting step; maybe more simple procedures are required to reach our ultimate goal – **getting robust nonlinear lattices for intense beams with self-consistent space charge distributions, immune to errors, mismatches and instabilities (like ep instability)**

Nonlinear KV- cont.

- **More important details**



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General thoughts on best distributions

- Why nonlinear lattice? Space charge provide tune spread itself. Unfortunately, instabilities (at SNS, PSR) don't profit from space charge spread – maybe because the centroid tune doesn't depend on space charge. Other reasons- mismatches, envelope oscillations near axis, etc. Also – new venue for AP

1) Our idea was to start with something very nonlinear and predictable and see what happens with perturbed beam (David's talk) with modest space charge and then increase it;

2) S. Lund idea is also in the same vein – to make adiabatic changes of intensity to see how self-consistent distribution evolves;

3) A. Sessler - Just take a periodic channel that is non-linear, put in some space charge and see if it settles down to something that is stable. I think this is what we should come to if we get very fast calculations – we need some Figure of Merit for final distributions to make good choice of initial distributions.

4) Many possibilities – one has a danger to sink into deep see of chaotic motion without understanding what is going on

3D case

- Nonlinear integrable traps for Laplacian fields were found (Nagaitsev, Danilov, http://arxiv.org/abs/1111.1260);
- Only some 3D stationary distributions and small deviations from them analyzed by Gluckstern, et al, PRE58, 4977 (1998)
- Really important for linacs like SNS one
- No 3D KV, but more involved linear self-consistent distributions exist (see Danilov, et al, 2003). Open question - 3d nonlinear motion with space charge

Conclusion

- Some examples of nonlinear resonance elimination (round beam and carb waist) successfully implemented;
- Nonlinear "integrable" accelerator optics is next step in the same direction;
- Examples of fully integrable motion exist for first ever implementations (IOTA ring), encouraging simulation results obtained by Tech-X;
- More solutions definitely exist – unfortunately, the mathematics is not well-developed for accelerators – it includes solving functional or high order partial differential equations;
- Virtually any next generation machine with nonlinearities can profit from resonance eliminations.